Düsseldorf Institute
for Competition Economics

DISCUSSION PAPER

No 285

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March 2018

HEINRICH HEINE
UNIVERSITÄT DÜSSELDORF

d|u|p düsseldorf university press

IMPRINT

DICE DISCUSSION PAPER

Published by

düsseldorf university press (dup) on behalf of
Heinrich-Heine-Universität Düsseldorf, Faculty of Economics,
Düsseldorf Institute for Competition Economics (DICE), Universitätsstraße 1,
40225 Düsseldorf, Germany
www.dice.hhu.de

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ISSN 2190-9938 (online) – ISBN 978-3-86304-284-4

Customer Recognition and Mobile Geo-Targeting

Irina Baye* Tim Reiz† Geza Sapi‡

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Abstract

We focus on four important features of mobile targeting. First, consumers' real-time locations are known to sellers. Second, location is not the only factor determining how responsive consumers are to discounts. Other factors such as age, income and occupation play a role, which are imperfectly observable to marketers. Third, sellers may infer consumer responsiveness from their past purchases. Fourth, firms can deliver personalized offers to consumers through mobile devices based on both their real-time locations and previous purchase behavior. We derive conditions that determine how combining behavior-based marketing with mobile geo-targeting influences profits and welfare in a competitive environment. Our setting nests some earlier models of behavior-based price discrimination as special cases and yields additional insights. For instance, different from previous studies we show that profit and welfare effects of behavioral targeting may depend on firm discount factor.

JEL-Classification: D43; L13; L15; M37.

Keywords: Mobile Marketing, Location Targeting, Price Discrimination, Customer Data.

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1 Introduction

The widespread use of smartphones revolutionized marketing by providing an advertising means that allows delivering personalized commercial messages depending on a wide range of customer characteristics. One of the most profitable and novel marketing opportunities opened up by mobile devices is geo-targeting. Various apps installed on a device read the GPS signals of users and share these with affiliated advertisers and retailers who can in turn send messages with commercial offers to users. Geo-targeted mobile advertising is booming. BIA Kelsey (2015) projects location-based mobile ad revenues in the U.S. to nearly triple within four years from $6.8 billion in 2015 to $18.2 billion in 2019. Mobile phone users differ also in other characteristics than location. Clearly, age, demographics, income, profession and many other factors influence how users respond to commercial offers and discounts. Mobile marketers routinely complement geo-location data with behavioral information on customers, which can signal their responsiveness to discounts (Thumbvista, 2015).

A prominent example recognizing both geo-location and behavioral data is the highly successful mobile marketing campaign of Dunkin' Donuts. In the first quarter of 2014, Dunkin' Donuts rolled out a campaign with discounts sent to phone users "*around competitors' locations coupled with behavioral targeting to deliver coupons on mobile devices*" (Tode, 2014).[1] The campaign proved highly lucrative, with a significant share of discount recipients showing interest and redeeming the coupon.

In our paper we focus on four important features of mobile targeting, which distinguish it from traditional targeting. First, consumers' real-time locations are known to sellers. Second, location is not the only factor determining how responsive consumers are to discounts. Other factors such as age, income and occupation play a role, which are imperfectly observable to marketers. Third, sellers may infer responsiveness from observing previous purchase behavior. Finally, firms can deliver personalized offers through mobile devices to individually addressable consumers. These features of mobile technology allow firms to engage in a new form of price discrimination by charging consumers different prices depending on both their real-time locations and previous purchase behavior. Our aim is to investigate how behavioral targeting

[1] Dunkin' Donuts complemented geo-location data with external data on behavioral profiles, obtained from billions of impressions gathered through mobile devices to identify anonymous Android and Apple device IDs. The campaign delivered banner ads to targeted devices that ran in the recipient's favorite apps or on mobile web sites. These ads featured offers such as a $1 discount on a cup of coffee and $2 discount on a coffee plus sandwich meal.

combined with perfect location-based marketing affects profits and welfare in a modeling setup that matches the main features of today's mobile marketing environment. We consider a model, where consumers differ along two dimensions: their *locations* and *flexibility*. We interpret location in a physical sense, as the focus of our paper is on mobile geo-targeting.[2] Flexibility is understood as the responsiveness of consumers to discounts. There are two firms, each selling a different brand of the same product and competing over two periods. We take as a starting point that consumer geo-locations are known to the firms. Additionally they can obtain behavioral information on the flexibility of customers by observing their purchases in the first period. To concentrate on the strategic effects of customer data, we assume that consumers are myopic while firms are forward-looking.

Our main results are as follows. First, we show that combining behavioral data with geo-targeting can influence second-period profits in three different ways, depending on how strongly consumers differ in their preferences. With weakly differentiated consumers, firms always gain from additional customer data and profits of the second period are the highest when this data is most precise. With moderately differentiated consumers, profits respond differently to behavioral data (depending on its quality) and firms are better off only if data is sufficiently accurate. Finally, when consumers are strongly differentiated among each other, firms are always worse off in the second period with behavioral data of any quality. The intuition reaches back to the standard insight in the price discrimination literature distinguishing between *rent-extraction* and *competition* effects of additional customer data. More data potentially allows firms to extract more rents from consumers, but targeting may also strengthen the intensity of price competition. When consumers are weakly differentiated, each firm wants to serve most consumers close-by. This, however, induces the rival to price aggressively even absent behavioral data. Firms therefore experience only the rent-extraction effect of additional data and their profits rise. When consumers are strongly differentiated, without behavioral data firms avoid tough competition by targeting consumers of different flexibility at each location. Precisely, each firm serves only the less flexible consumers among those located closer to it. Additional customer data intensifies competition because firms compete for each group previously served by only one firm, which makes both of them worse off. Finally, when consumers are moderately differentiated, the profit effect of behavioral data in the second period depends on the interplay between the rent-

[2] Our results would apply equally if location were interpreted in *preference space*, as is often done with spatial models of product differentiation.

extraction and competition effects. In this case firms are better off only if data is sufficiently precise.

Second, we find that firms strategically influence the quality of the (revealed) behavioral data by choosing the respective first-period prices. This is due to the fact that data quality is interlinked with the distribution of firm market shares in the first period and data is most precise when on a given location firms can distinguish between two consumer groups of equal size. In addition, the value of behavioral data to the firms depends on the strength of consumer heterogeneity. When consumers are relatively homogeneous, the value of additional flexibility data is low. In this case every firm serves all consumers close-by in the initial period, such that no data about their flexibility is revealed in equilibrium. With more differentiated consumers, behavioral data boosts profits in the second period and its quality in equilibrium is higher when firms value future profits more. In contrast, with strongly differentiated consumers flexibility data intensifies competition in the second period. Consequently, firms end up with less precise information when they discount future profits less. Overall, firms influence the quality of the revealed behavioral data in a way that allows them to realize the highest profits over two periods. Precisely, when they expect the rent-extraction effect to dominate, they make sure to gather more precise information about customers. When data intensifies competition, firms strategically distort its quality downwards.

Third, we compare the overall profits with the situation where firms cannot collect behavioral data for some exogenous reasons. We first isolate cases where the profit effect of data can be assessed in a quite simple manner, as it is driven purely by the level of consumer differentiation. In particular, if consumers are relatively homogeneous, the value of behavioral data is low and firms do not collect it in equilibrium, such that the ability to observe consumers' past purchases does not influence profits. In contrast to conventional wisdom, firms are worse off when they can collect behavioral data on consumers if these are very different. In turn, behavioral data boosts the overall profits if consumers are moderately differentiated, because in this case additional data does not intensify competition dramatically. Finally, if consumer differentiation is in between these *pure* cases the effect of behavioral data on profits depends on the discount factor and is likely to be positive when firms value future profits more. In this case firms have stronger incentives to adjust their price choices in the initial period, which increases the overall profits. We also show that consumers' and firms' interests are not necessarily opposed. If consumers are moderately differentiated, both consumer surplus and profits can increase when firms are able to observe customers' past purchases leading to a higher social welfare.

2 Related Literature

Our paper contributes to two main strands of literature. The first strand analyzes the competition and welfare effects of firms' ability to recognize past customers and price discriminate between them and the rival's customers in the subsequent periods.[3] The second relatively new and actively developing strand analyzes competition between mobile marketers who can observe geo-locations of consumers and target them with personalized offers.

In the literature on behavior-based targeting firms adjust their prices in the first period taking into account the impact of behavioral data on their future profits. Corts (1998) proposes an elegant way how to predict the price and profit effects of firms' ability to discriminate among two customer groups. He distinguishes between two types of markets. If for a given uniform price of the rival both firms optimally charge a higher price to the same consumer group, then according to Corts such market is characterized by *best-response symmetry*. In all other cases *best-response asymmetry* applies. Corts shows that with best-response asymmetry targeted prices to both consumer groups change in the same direction relative to the uniform price. This change may be either positive or negative leading to higher or lower discrimination profits, respectively. Thisse and Vives (1988) were the first to demonstrate the negative effect of price discrimination on prices and profits leading to a prisoners' dilemma situation.[4] More recent literature showed that firms may be better off with price discrimination under best-response asymmetry.[5]

The seminal article by Fudenberg and Tirole (2000) considers a dynamic Hotelling-type duopoly model with horizontally differentiated consumer preferences and a market showing best-response asymmetry. In the initial period firms quote uniform prices, while in the subsequent period they can offer different prices to the former customers of each other. Under uniformly distributed consumer preferences second-period profits always remain below the level without

[3] Fudenberg and Villas-Boas (2006) provide a review of this literature.

[4] For a similar result see Shaffer and Zhang (1995), Bester and Petrakis (1996) and Liu and Serfes (2004).

[5] Precisely, the positive profit effect is demonstrated in articles starting with an asymmetric (more advantageous to one of the firms) situation (see Shaffer and Zhang, 2000 and 2002; Carroni, 2016) and in articles which assume imperfect customer data (as in Chen *et al.*, 2001; Liu and Shuai, 2016; Baye and Sapi, 2017).

price discrimination.[6] This is so because customer data intensifies competition: A consumer's purchase at a respective firm in the first period reveals her (relative) preference for that firm's product making the rival compete aggressively for such consumers, which creates a downward pressure for this firm's profits. Interestingly, Fudenberg and Tirole find that second-period considerations do not influence the first-period equilibrium, as a result firms end up with the lowest second-period profits possible and are worse off being able to recognize own customers.[7,8]

Esteves (2010) adopts a discrete distribution of customer preferences. In her model there are two consumer groups who prefer the product of a given firm if its price does not exceed that of the rival by more than a given amount (referred to as "the degree of consumer loyalty"). Esteves investigates both the static and dynamic (two-period) games for an equilibrium in mixed strategies and considers myopic consumers. Similar to Fudenberg and Tirole (2000), in Esteves firms are also better off in the second period when they cannot engage in behavioral price discrimination. However, second-period profit considerations do influence the prices of the first period. Firms price to soften future competition and consequently the probability of an outcome where they learn consumer preferences decreases when firms become more patient.

Chen and Zhang (2009) consider a market with three consumer segments, two of which are price-insensitive consumers who always purchase from the preferred firm. The third segment consists of switchers who buy from the firm with a lower price. The authors solve the model for an equilibrium in mixed strategies and assume that both firms and consumers are forward-looking. Surprisingly, profits when firms are unable to collect behavioral data are equal to those with automatic customer recognition. However, firms are better off if they must actively gather customer data. The reason is that a firm which is able to recognize its loyal consumers benefits from the rent-extraction in the second period. As a result, each firm charges a relatively high price in the first period to separate its loyal consumers from switchers, thereby softening competition.[9]

[6] Fudenberg and Villas-Boas (2006) provide a detailed analysis of the uniform case of Fudenberg and Tirole (2000).

[7] This result is driven by the fact that the profits of the second period get their minimum at equal market shares, such that the optimal prices following from the first-order conditions in a one-period and dynamic models coincide.

[8] In related articles Villas-Boas (1999) and Colombo (2016) also show that firms are worse off with the ability to recognize consumers. Villas-Boas derives this result in a model with infinitely lived firms and overlapping generations of consumers, while Colombo assumes that firms can recognize only a share of their previous customers.

[9] In Chen and Zhang (2009) customer data is fully revealed in equilibrium. Precisely, the firm with a higher price in the first period identifies all of its loyal consumers, because none of them foregoes a purchase to pretend

The model proposed in this paper nests the setups of Fudenberg and Tirole (2000), Esteves (2010) and Chen and Zhang (2009) as special cases. In particular, we obtain the results of Fudenberg and Tirole and Esteves when consumers are strongly differentiated. Firms are then worse off with additional customer data irrespective of the discount factor. In Esteves, with customer data firms can discriminate between two groups of consumers loyal to one of the firms. In contrast, in our analysis customer data allows to distinguish at any geo-location among two groups with different flexibility *within* the loyal consumers of a firm.[10] It follows that compared to our model, the level of consumer differentiation considered by Esteves is higher.[11] Also similar to Esteves, we show that with strongly differentiated consumers firms choose first-period prices so as to minimize the quality of the revealed customer data.

We also get the result of Chen and Zhang (2009) where firms are better off with behavioral targeting irrespective of the discount factor when we consider weakly differentiated consumers. The reason is that in Chen and Zhang firms compete for price-sensitive switchers who always buy at a firm with a lower price and are, hence, fully homogeneous in their preferences. We conclude that the level of consumer differentiation in preferences can serve as a reliable tool for predicting profit effects of targeted pricing based on customer behavioral data.[12] A further novelty of our paper compared to the previous studies is to demonstrate that this effect may also depend on the discount factor and behavioral targeting is more likely to boost profits if firms value future profits more. All previous articles on behavior-based price discrimination known to us found that it increases or reduces profits, irrespective of the discount factor.

We also contribute to the rapidly growing literature on oligopolistic mobile geo-targeting. Chen *et al.* (2017) consider a duopoly model with consumers located at one of the two firms' addresses as well as some consumers situated in the middle between the firms. The authors assume that consumers are differentiated along two dimensions: locations and brand preferences. Furthermore, some consumers are aware of available offers at different locations and choose

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to be a switcher.

[10] In our analysis, consumers are loyal to a firm if they buy from it when both firms charge equal prices.

[11] Slightly different from Esteves (2010), in Fudenberg and Tirole (2000) with behavioral data firms can identify two consumer groups, which are only *on average* more loyal to one of them. However, this also implies a high level of consumer heterogeneity than in our analysis.

[12] Our result that behavior-based targeting is more likely to be profitable if consumers are less differentiated, depends on the symmetry of customer data available to the firms. Shin and Sudhir (2010) show that it can be reversed when customer data is asymmetric. Precisely, in their analysis every firm can distinguish between the own low- and high-demand customers of the previous period, while the rival only knows that these consumers bought from the other firm.

among these, but incur travel costs. Chen *et al.* show that mobile targeting can increase profits compared to the uniform pricing, even in the case where traditional targeting (where consumers do not seek for the best mobile offer) does not. Unlike in Chen *et al.*, in our model firms start out with mobile geo-targeting ("traditional targeting," according to Chen *et al.*) and we analyze how the ability to collect additional behavioral data influences profits. Different from Chen *et al.*, we also vary the level of consumer heterogeneity in preferences and show that it is crucial for predicting the profit and welfare effects of combining behavior-based pricing with mobile geo-targeting.[13]

Dubé *et al.* (2017) conduct a field experiment to analyze the profitability of different mobile marketing strategies in a competitive environment. In their analysis two firms can target consumers based on both their locations and previous behavior. They show that in equilibrium firms choose to discriminate only based on consumer behavior, in which case profits increase above those with uniform prices. However, profits would be even higher if firms applied finer targeting, which relies on both behavioral and location data. Our paper differs from Dubé *et al.* in two important ways. First, we show that customer targeting based on both location and behavioral data does not necessarily increase profits and can harm firms if consumers are sufficiently differentiated.[14] Second, in our model behavioral data is generated endogenously in a dynamic setting, where firms strategically influence its quality. For instance, when consumers are rather homogeneous, in equilibrium firms do not collect any behavioral data unless they are quite patient, although they would benefit from such data in the second period.[15]

Finally, Baye and Sapi (2017) also consider today's mobile marketing data landscape, where firms can use near-perfect customer location data for targeted pricing. They analyze how firms' incentives to acquire (costlessly) data on other consumer attributes being (imperfect) signals of their flexibility depend on its quality. In our model firms collect additional data through observing consumers' purchase histories, which is not costless, because firms have to sacrifice

[13] To keep our analysis tractable we do not allow consumers to strategically change their locations (to get the best mobile offer) and assume that they are targeted at their home locations. We also find this level of consumer sophistication realistic in markets where our analysis applies most.

[14] Interestingly, when commenting why geo-targeting is more profitable than behavioral targeting, Dubé *et al.* (2017) explain it through the difference in the levels of consumer heterogeneity over locations and purchase behavior (recency).

[15] Dubé *et al.* (2017) also recognize that in a dynamic setting customer segmentation (derived from the accumulated customer data) is determined endogenously. They argue that "*an interesting direction for future research would be to explore how dynamics affect equilibrium targeting and whether firms would continue to profit from behavioral targeting.*"

some of their first-period profits to gain customer data of a better (worse) quality. As a result, compared to our paper, Baye and Sapi overestimate both the benefit of additional customer data to firms and its damage to consumers in mobile marketing.

3 The Model

There are two firms, A and B, that produce two brands of the same product at zero marginal costs and compete in prices. They are situated at the ends of a unit Hotelling line: Firm A is located at $x_A = 0$ and firm B at $x_B = 1$. There is a unit mass of consumers each with an address $x \in [0, 1]$ on the line, which describes her real physical location, as transmitted by GPS signals to retailers in mobile marketing. If a consumer does not buy at her location, she incurs linear transport costs proportional to the distance to the firm. We follow Jentzsch *et al.* (2013) and Baye and Sapi (2017) to assume that consumers differ not only in their locations, but also in transport costs per unit distance (flexibility), $t \in \left[\underline{t}, \bar{t}\right]$, where $\bar{t} > \underline{t} \geq 0$.[16,17] Transport costs are higher if t is larger. Each consumer is uniquely characterized by a pair (x, t). We assume that x and t are uniformly and independently distributed giving rise to the following density functions: $f_t = 1/(\bar{t} - \underline{t})$, $f_x = 1$ and $f_{t,x} = 1/(\bar{t} - \underline{t})$. The utility of a consumer (x, t) from buying at firm $i = \{A, B\}$ is

$$U_i(p_i(x), t, x) = v - t\,|x - x_i| - p_i(x). \tag{1}$$

In equation (1) $v > 0$ denotes the basic utility, which is assumed high enough such that the market is always covered in equilibrium. A consumer buys from the firm whose product yields higher utility.[18] Without loss of generality, we normalize $\bar{t} = 1$ and measure the level of consumer heterogeneity by the ratio of the largest to the lowest transport cost parameter: $l := \bar{t}/\underline{t} = 1/\underline{t}$,

[16] Different from Jentzsch *et al.* (2013) and Baye and Sapi (2017), data on consumer flexibility is generated endogenously in our model and, as we show below, firms influence strategically the quality of the revealed customer data.

[17] Esteves (2009), Liu and Shuai (2013 and 2016), Won (2017) and Chen *et al.* (2017) also consider a market, where consumer preferences are differentiated along two dimensions. However, in their analysis the strength thereof (flexibility) is the same among all consumers. In Borenstein (1985) and Armstrong (2006) consumers also differ in their transport cost parameters. Both show that firms may benefit from discrimination along this dimension of consumer preferences. In our analysis where firms are endowed with the ability to target consumers based on their locations, the profit effect of behavioral data on consumer flexibility depends on the level of the overall consumer heterogeneity and firm discount factor.

[18] We follow the tie-breaking rule of Thisse and Vives (1988) and assume that if a consumer is indifferent, she buys from the closer firm. If $x = 1/2$, then in the case of indifference a consumer buys from firm A.

with $l \in (1, \infty)$. As we show below, parameter l plays a crucial role in our analysis. To consumers with $x < 1/2$ ($x > 1/2$) located closer to firm A (firm B) we refer as the *turf* of firm A (B). We also distinguish among consumers at the same location. Precisely, we refer to consumers with lower (higher) transport cost parameters as more (less) flexible ones.

We assume that firms observe with perfect precision the physical locations of all consumers in the market. There are two periods in the game. In the first period location is the only dimension by which firms can distinguish consumers. They issue targeted offers at the same time to all consumers, depending on their locations. Consumers at the same location will receive the same targeted offer. In the second period firms again send simultaneously targeted offers to consumers. However, this time they are able to distinguish among consumers that visited them in the first period and those that did not.[19] As a result, in the second period firms can charge (up to) two different prices at each location: one to the own past customers and the other one to those of the rival. Table 1 summarizes the three types of information firms can obtain in our model. We analyze how this information translates into pricing decisions in a dynamic competitive environment. We assume that firms are forward-looking while consumers are myopic, which allows us to concentrate on the strategic effects of customer data.[20] We solve for a subgame-perfect Nash equilibrium and concentrate on equilibria in pure strategies.

Type of customer data	Time obtained	Quality
Geo-location	Real-time in each period	Perfect
Own/rival's past customer	Inferred from 1-st period purchasing decisions	Perfect
Flexibility	Inferred from 1-st period purchasing decisions	Imperfect

Table 1: Customer data available to the firms

[19] Danaher *et al.* (2015) show in a field experiment, where all consumers got coupons with the *same* discount, that both the consumer's distance to the store and her previous behavior (redemption history) determine the probability that a coupon will be redeemed by a customer. It is then consistent with these results that in our model where firms can target consumers with *personalized* coupons, they use the information on both customer locations and their purchase history to design coupons. Similarly, Luo *et al.* (2017) show in a recent field experiment that depending on consumer locations different temporal targeting strategies are needed to maximize consumer responses to mobile promotions. This also speaks for a necessity to target consumers individually depending on location.

[20] See Esteves (2010) for a similar assumption.

4 Equilibrium Analysis

We start from the second period, where firms can discriminate depending on both consumer locations and their behavior.

Equilibrium analysis of the second period. As firms are symmetric, it is sufficient to analyze a single location for instance on firm A's turf. Consider an arbitrary $x < 1/2$. Let t^α denote the transport cost parameter such that consumers with $t \geq t^\alpha$ visited firm A in the previous period, while consumers with $t < t^\alpha$ purchased at firm B.[21] If the share of consumers at x who bought from firm B in the first period is $\alpha \in [0, 1]$, then $t^\alpha := \alpha + \underline{t}\,(1 - \alpha)$. We will refer to consumers with $t < t^\alpha$ as segment α and to those with $t \geq t^\alpha$ as segment $1 - \alpha$. Segment α includes the relatively flexible consumers who purchased in the first period from the firm located further away. We denote the prices of the second period to consumers on segments α and $1 - \alpha$ as p_i^α and $p_i^{1-\alpha}$, with $i = A, B$, respectively. Firms choose the prices so as to maximize their profits on each segment separately. Figure 1 depicts both segments at location x.

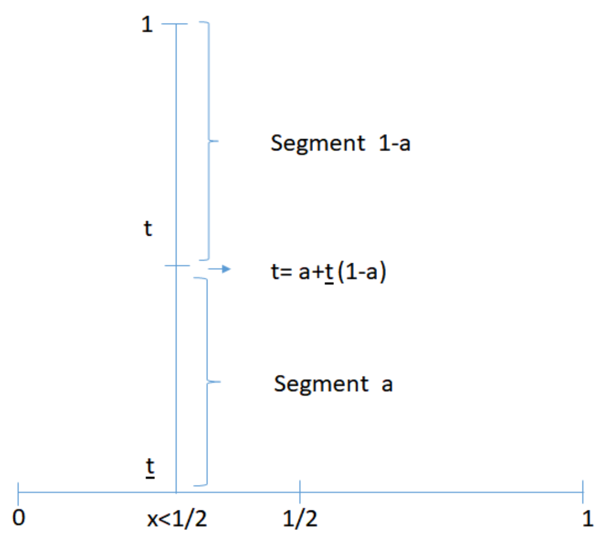

Figure 1: Segments α and $1 - \alpha$ at some location $x < 1/2$

Consider segment α. On its own turf firm A can attract consumers with sufficiently high

[21] A standard revealed-preference argument implies that if a consumer on $x < 1/2$ with $t = \widetilde{t}$ bought from firm A (B) in the first period, then all consumers with $t > \widetilde{t}$ ($t < \widetilde{t}$) made the same choice.

transport cost parameters, such that

$$U_A(p_A^\alpha(x),t,x) \geq U_B(p_B^\alpha(x),t,x) \text{ implies } t \geq t_c^\alpha\left(p_A^\alpha(x),p_B^\alpha(x)\right) := \tfrac{p_A^\alpha(x)-p_B^\alpha(x)}{1-2x}.$$

Firms choose prices $p_A^\alpha(x)$ and $p_B^\alpha(x)$ to maximize their expected profits:

$$\max_{p_A^\alpha(x)} \tfrac{[t^\alpha - t_c^\alpha(\cdot)]p_A^\alpha(x)}{1-\underline{t}} \text{ and } \max_{p_B^\alpha(x)} \tfrac{[t_c^\alpha(\cdot)-\underline{t}]p_B^\alpha(x)}{1-\underline{t}}.$$

The mechanism is analogous on segment $1-\alpha$. The following lemma describes the equilibria on each segment at a given location $x < 1/2$ depending on market shares of the first period.[22]

Lemma 1. *Consider an arbitrary x on the turf of firm A. The equilibrium on each segment at this location depends on the asymmetry between first-period market shares.*
i) If in the first period firm B's market share at this location was low, $\alpha \leq 1/(l-1)$, then it attracts no consumer on segment α in the second period, where firm A charges the price $p_A^\alpha(x) = \underline{t}(1-2x)$ and the price of firm B is zero. Otherwise, firm B serves the more flexible consumers on segment α, with $t < \underline{t}\left[\alpha(l-1)+2\right]/3$ at the price $p_B^\alpha(x) = \underline{t}(1-2x)\left[\alpha(l-1)-1\right]/3$, while the price of firm A is $p_A^\alpha(x) = \underline{t}(1-2x)\left[2\alpha(l-1)+1\right]/3$.
ii) If in the first period firm B's market share at this location was high, $\alpha \geq (l-2)/\left[2(l-1)\right]$, then it attracts no consumer on segment $1-\alpha$, where firm A charges the price $p_A^{1-\alpha}(x) = t^\alpha(1-2x)$ and the price of firm B is zero. Otherwise, firm B serves the more flexible consumers on segment $1-\alpha$, with $t < \underline{t}\left[l+1+\alpha(l-1)\right]/3$, and firms charge prices $p_A^{1-\alpha}(x) = \underline{t}(1-2x)\left[2l-1-\alpha(l-1)\right]/3$ and $p_B^{1-\alpha}(x) = \underline{t}(1-2x)\left[l-2-2\alpha(l-1)\right]/3$.

Remember that α denotes the share of consumers at location x that bought from firm B in the previous period. If α is small, then consumers on segment α are relatively similar in flexibility, such that firm A can attract them all without having to significantly reduce the price targeted at the least flexible consumer (with $t = t^\alpha$). As a result, the monopoly equilibrium emerges on segment α where firm A serves all consumers and firm B cannot do better than charging zero. Analogously, if α is large then the complementary segment $1-\alpha$ is relatively small, such that in equilibrium firm A serves all consumers there. In contrast, with large α, consumers are quite different in their preferences on segment α. In this case firm A prefers to extract rents from the less flexible consumers there and lets the rival attract the more flexible

[22] All the omitted proofs are contained in Appendix.

ones. The following lemma characterizes the equilibria at any location $x < 1/2$ depending on the heterogeneity parameter, l, and the first-period market share of firm B, α.

Lemma 2. *Consider an arbitrary x on the turf of firm A. The equilibrium at this location depends on the asymmetry between first-period market shares and consumer heterogeneity in flexibility.*
i) If $l \leq 2$, then firm A attracts all consumers at x irrespective of α.
ii) If $2 < l < 4$, then firm A attracts all consumers at x provided firm B's first-period market share at this location was intermediate, i.e., $1/(l-1) \leq \alpha \leq (l-2)/[2(l-1)]$. Otherwise, firm A loses consumers on one of the segments.
iii) If $l \geq 4$, then irrespective of α firm A loses some consumers at x. However, it can monopolize segment α $(1-\alpha)$ provided firm B served in the first period relatively few (many) consumers at that location, with $\alpha \leq 1/(l-1)$ $(\alpha \geq (l-2)/[2(l-1)])$. For intermediate values of α, $1/(l-1) < \alpha < (l-2)/[2(l-1)]$, both firms serve consumers on both segments.

If $l \leq 2$, then independently of firm B's first-period market share consumers on both segments are quite similar in their preferences, such that in equilibrium firm A serves all consumers at x for any α. When consumers become more differentiated, with $2 < l < 4$, the optimal strategy of firm A depends on the market share of firm B in the first period. Precisely, if α takes intermediate values, then consumers have similar flexibility on each segment yielding again monopoly equilibria on both segments. Finally, for $l \geq 4$ irrespective of α on each segment firm A faces very different consumers and always loses some of them in equilibrium. Figure 2 provides two examples of the second-period equilibrium at some $x < 1/2$ depending on l and α and shows each firm's demand regions on both segments.

We next analyze how total profits in the second period change with α. We assume that every firm served the share α of customers at any location on the rival's turf in the first period.[23] The following proposition summarizes our results.

Proposition 1. *Assume that in the first period each firm served the share α of consumers at any location on the rival's turf. A firm's second-period profits as a function of α depend on how strongly consumers differ in their preferences.*

[23] We make this assumption to derive each firm's total profits (at all locations together) in the second period in order to compare them with the similar profits from the other relevant studies mentioned above. Moreover, we demonstrate below that the first-period market share of each firm, α, in equilibrium is indeed the same at any location on the rival's turf.

i) If $l \leq 2.38$, profits are an inverted U-shaped function of α. Moreover, for any $\alpha \in (0,1)$ profits are higher than at $\alpha = 0$ ($\alpha = 1$). The highest profit level is attained at $\alpha = 1/2$.
ii) If $2.38 < l < 8$, profits are given by different non-monotonic functions of l, sharing the following common features: First, there exists $\widehat{\alpha}(l)$, such that profits are lower than at $\alpha = 0$ ($\alpha = 1$) if $\alpha < \widehat{\alpha}(l)$ and are higher otherwise. Moreover, $\partial\widehat{\alpha}(l)/\partial l > 0$. Second, the highest profit level is attained at $\alpha = 1/2$ if $l < 2.8$ and at $\alpha = (9l - 16) / [8(l-1)]$ otherwise.
iii) If $l \geq 8$, profits are a U-shaped function of α. Moreover, for any $\alpha \in (0,1)$ profits are lower than at $\alpha = 0$ ($\alpha = 1$). The lowest profit level is attained at $\alpha = (2l - 3) / [5(l-1)]$.

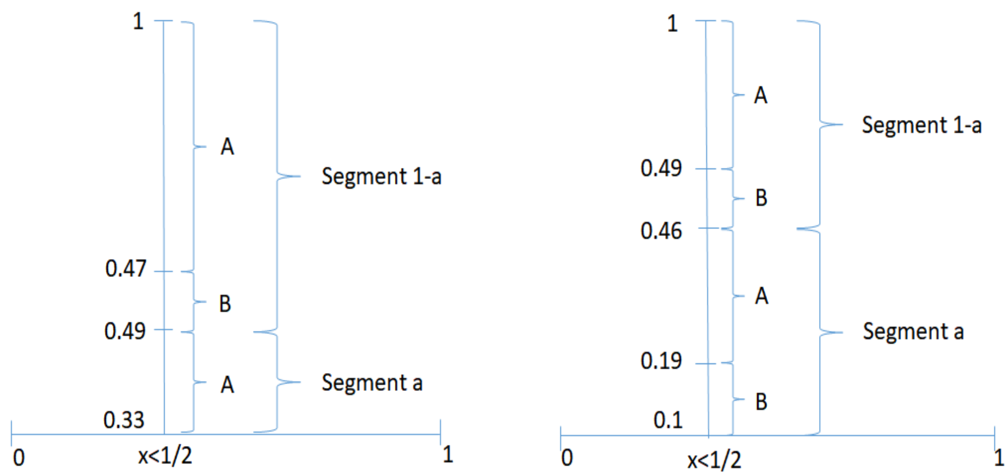

Figure 2: Demand regions at some $x < 1/2$ in the second period for $l = 3$ and $\alpha = 0.2$ (left) and $l = 10$ and $\alpha = 0.4$ (right)

The impact of combining behavioral data with geo-targeting on profits of the second period is driven by two effects: rent-extraction and competition. Precisely, with behavioral data every firm can recognize its past customers and distinguish between these and the more flexible ones who bough from the rival. It then charges higher prices to the former, which describes the rent-extraction effect. In contrast, the rival targets more aggressively exactly these consumers, to which we refer as the competition effect. The overall effect of additional data on profits depends on the interplay between these two opposing effects and is driven by the ratio l and the quality of the gained data. In the extreme cases of $\alpha = 0$ or $\alpha = 1$, flexibility data does not provide any additional information on customer preferences, because all consumers at a given location bought from the same firm in the first period. In contrast, the highest level of data accuracy is attained when the segments are of equal size, i.e., at $\alpha = 1/2$.[24] Figures 3a and 3b depict firm profits in the second period as a function of α for all the three cases described in Proposition 1 with $l = 2$, $l = 3$ and $l = 10$, respectively.

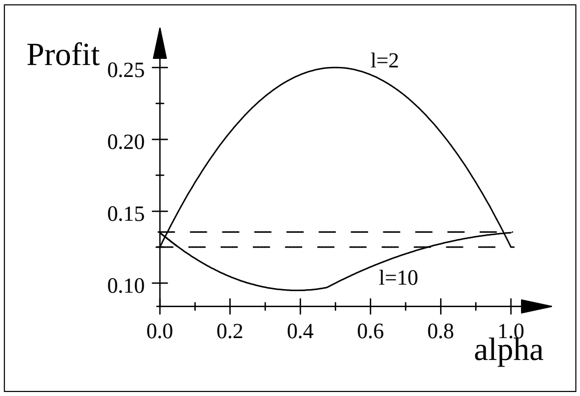

Figure 3a: Profits of the second period depending on the market share of firm B in the first period, α ($l = 2$ and $l = 10$)

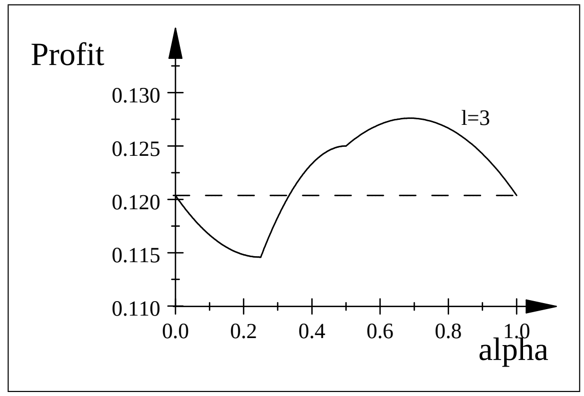

Figure 3b: Profits of the second period depending on the market share of firm B in the first period, α ($l = 3$)

According to Lemma 2, if $l \leq 2$, then in the second period each firm serves all consumers at any location on its own turf independently of how consumers were distributed among the firms in the first period.[25] The rival cannot do better than charging zero on both segments for any α, such that the competition effect of flexibility data is absent. The remaining rent-extraction effect is in turn strongest when data precision is the highest, with $\alpha = 1/2$.

Unlike in the previous case where the changes in profits with α were driven purely by the rent-extraction effect, in the case of $2.38 < l < 8$ it is the interplay of the two effects, which determines how profits change. As α increases above $\alpha = 0$, firms get some additional data from consumers' purchase histories. This in turn boosts competition strongly: The rival decreases

[24] It is easy to see this also formally. We can say that data is most precise if it allows a firm to extract the highest rents from the consumers located closer to it on a given address for any price of the rival. The latter condition allows to abstract from the competition effect. Consider some x on the turf of firm A and let the rival's price be p_B. To serve all consumers on segments α and $1-\alpha$ firm A has to charge prices $p_A^{\alpha}(x) = \underline{t}\,(1-2x) + p_B$ and $p_A^{1-\alpha}(x) = (\alpha + \underline{t}\,(1-\alpha))\,(1-2x) + p_B$, respectively, which yield the total profit of firm A on location x: $\Pi_A\,(x) = p_B + (1-2x)\,\underline{t}\left[(1-\alpha)^2 + \alpha l\,(1-\alpha) + \alpha\right]$. This profit gets its maximum $\alpha = 1/2$ for any l and any p_B. Similarly, Fudenberg and Villas-Boas (2006) in their analysis of Fudenberg and Tirole (2000) also argue that data is most precise when firms can distinguish between two consumer groups of equal size.

[25] While in the case of $2 < l \leq 2.38$, any firm loses some consumers at any location on its turf in equilibrium of the second period, total profits over both turfs behave in the same ways as in the case $l \leq 2$.

its prices on both segments, eroding profits. When data quality improves further, the rent-extraction effect starts to take over and profits increase. Overall, profits are the highest when data is more accurate, i.e., α takes intermediate values (close to $\alpha = 1/2$). With a further increase in α, behavioral data becomes less and less precise about the consumers who bought from the rival in the initial period decreasing its overall predictive power and profits altogether. However, profits then always remain above the level without flexibility data (at $\alpha = 1$ or $\alpha = 0$).

If $l \geq 8$, profits drop rapidly as α becomes strictly positive. As a result, although profits start recovering when α increases above a certain threshold, they never exceed the level without behavioral data (at $\alpha = 1$ or $\alpha = 0$). Interestingly, in this case profits are the lowest when data is most precise (α takes intermediate values), because competition is most intense then. The case $l \geq 8$ is similar to the result obtained by Fudenberg and Tirole (2000) for the uniformly distributed consumer preferences. This similarity is driven by the fact that their model corresponds to the case of very high l in our analysis. Indeed, behavioral customer data in Fudenberg and Tirole allows to distinguish among two consumer groups loyal (on average) to *different* firms, while in our setting at each location in the second period firms can discriminate among two consumer groups loyal to the *same* firm. We now turn to the analysis of the first period.

Equilibrium analysis of the first period. In this subsection we analyze competition in the first period where firms can discriminate only based on consumer locations and charge prices to maximize their discounted profits over two periods. Similar to Fudenberg and Villas-Boas (2006) we concentrate only on equilibria in pure strategies in the first period.[26] The proposition below summarizes our results.[27]

Proposition 2. *Consider an arbitrary location x on the turf of firm $i = A, B$. The subgame-perfect Nash equilibrium (in pure strategies) takes the following form:*

i) First period. In equilibrium firm i monopolizes location x only if consumers are relatively homogeneous, i.e., $l \leq h_1(\delta)$, with $h_1(0) = 2$, $h_1(1) = 1.5$ and $\partial h_1(\delta)/\partial\delta < 0$. Otherwise, in the first period firms share consumers at x, such that the more flexible of them buy at the more distant firm.

[26] If $2 < l < 2,89$ or $5 < l < 14,13$ there are some values of firm discount factor, for which the equilibrium in pure strategies in the first period does not exist or two equilibria in pure strategies exist. In Proposition 2 we consider only those constellations of parameters l and δ, which yield the unique equilibrium prediction in pure strategies in the first period. This becomes more likely when firms are less patient because in that case the dynamic maximization function is close to the static one.

[27] In Proposition 2, "h" stays for the critical levels of consumer *heterogeneity*.

ii) Second period. In equilibrium firm i monopolizes location x if consumers are weakly differentiated, i.e., $l \leq h_2(\delta)$, with $h_2(0) = 2$, $\partial h_2(\delta)/\partial\delta > 0$ and $h_2(\delta) > h_1(\delta)$ for any $\delta > 0$. In equilibrium firm i serves all consumers on segment α, while the more flexible consumers on segment $1-\alpha$ buy at the rival provided consumers are moderately differentiated, i.e., $h_3(\delta) \leq l \leq \min\{h_4(\delta), h_5(\delta)\}$, with $h_3(0) = 2$, $h_n(0) = 5$, $\partial h_n(\delta)/\partial\delta > 0$ and $h_n(\delta) > h_3(\delta)$ for any δ, $n = 4, 5$. Finally, if consumers are strongly differentiated, i.e., $l \geq \max\{h_4(\delta), h_5(\delta)\}$, then firm i serves the less flexible consumers on both segments, while the more flexible consumers buy at the rival, with $\max\{h_4(1), h_5(1)\} = 14.13$.

The subgame-perfect Nash equilibrium in pure strategies is driven by both the level of consumer differentiation and firm discount factor. Although the relationship is intertwined, there are parameter ranges that allow unambiguous insights. In particular, if $l \leq 1.5$, then in equilibrium each firm serves all customers at any location on its turf in both periods. If $2.89 \leq l \leq 5$, then in equilibrium each firm attracts only the less flexible consumers close by in the first period, while in the following period all of them as well as the more flexible consumers on segment $1-\alpha$ buy at that firm. Finally, if $l \geq 14.13$, then in equilibrium each firm loses the more flexible consumers in the first period and also on any segment in the second period. Proposition 2 states that for other values of consumer differentiation in flexibility, the equilibrium depends on how strongly firms value future profits. In this case a sufficiently high discount factor leads to the monopoly outcome in the second period (on one or both segments). Figure 4 depicts the critical values of l (as a function of δ), which give rise to the equilibria stated in Proposition 2.

We observe from Proposition 2 that with increasing consumer heterogeneity (l gets larger), the equilibrium where firms lose some of the close-by consumers becomes more likely in the second period. This conclusion allows us to qualify the results of Lemma 2, which yields multiple equilibrium predictions for $l > 2$ depending on α. Precisely, it establishes that in equilibrium of the second period a firm serves all consumers on segment $1-\alpha$ at any location on its turf and at the same time loses some consumers on the complementary segment if the share α is large enough. As Proposition 2 shows, this outcome never emerges on the equilibrium path. It is useful to recall that segment α includes those consumers at some location on a firm's turf who bought from the rival in the previous period. As none of the firms loses in the equilibrium of the initial period more than half of the consumers on its turf, this segment is relatively small, which makes it easy for a firm to monopolize it in the subsequent period relying on the acquired behavioral data.

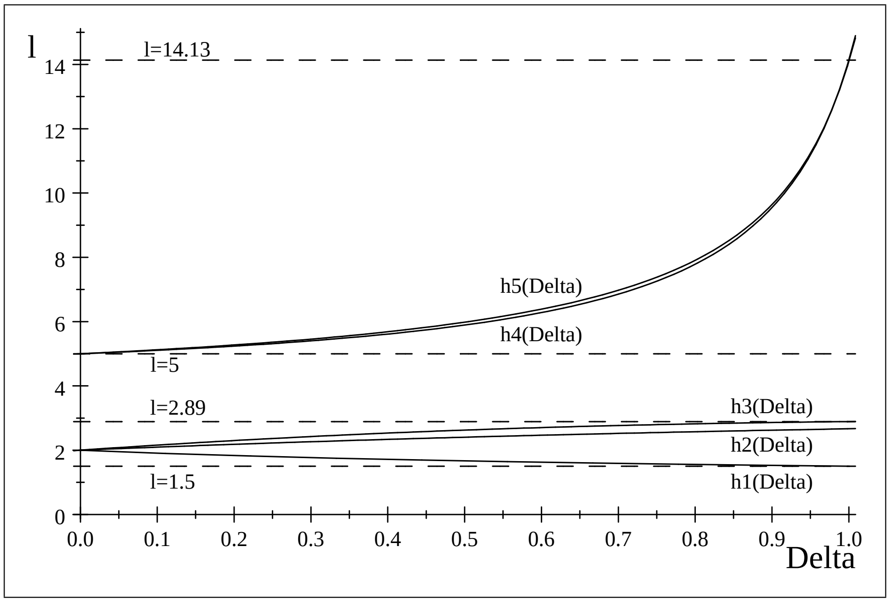

Figure 4: Critical values of parameter l giving rise to the equilibria stated in Proposition 2

The equilibrium of the first period also depends on how strongly consumers differ among each other. In particular, firms serve all consumers on their turfs if these are extremely similar ($l \leq h_1(\delta)$). Otherwise, they prefer to attract the less flexible consumers located close by with a correspondingly high price, allowing the flexible ones to buy at the rival. Unlike in the subsequent period, in the initial period firms additionally take into account the dynamic effect of their pricing decisions on future profits. The allocation of consumers in the first period determines the quality of information about their flexibility to be used in the future.

To understand how dynamic considerations influence pricing decisions, it is useful to start with the case of $\delta = 0$, where firms are short-sighted and fully ignore future profits. They serve all consumers on their turfs if $l \leq 2$ and lose the more flexible ones to the rival otherwise. Comparing this result with the case where firms value future profits ($\delta > 0$), we observe that the dynamic effect is absent when consumers are *relatively homogeneous* ($l \leq 1.5$): Firms monopolize any location on their turfs both with $\delta = 0$ and any $\delta > 0$. The reason is that when consumers are similar in their preferences, the value of the additional customer data is low and firms optimally prefer not to distort their pricing decisions of the first period. Note further that with $l \leq 1.5$ the dynamic effect is absent independently of δ. However, as differentiation becomes stronger, with $1.5 < l \leq 2$, discounting starts to play a role and first-period pricing decisions remain undistorted by future profits considerations only if δ is sufficiently small. Otherwise, firms sacrifice some of the short-run profits to be able to extract higher rents in the future: They lose some consumers at any location on their turfs in order to gain additional data about their preferences. Overall, the dynamic considerations play a role if consumer differentiation is sufficiently strong and enough weight is put on second-period profits in discounting.

If $l > 2$, the sharing equilibrium prevails in the first period with any $\delta \geq 0$, such that firms always gain some behavioral customer data. To understand how dynamic considerations drive pricing decisions in this case, we analyze how the distribution of consumers at a given location depends on the discount factor. We focus on the derivative of $\alpha^* (\delta)$, the market share of the rival at some location on a firm's turf in the first period, with respect to δ. If $\partial \alpha^* (\delta) / \partial \delta > 0$, we conclude that more (better) customer data is revealed in the first period when firms become more patient. Since the share $\alpha^* (\delta)$ always includes less than half of consumers at any location, a larger $\alpha^* (\delta)$ implies a more symmetric distribution of consumers between the firms and, hence, more information gained about their preferences. The following corollary summarizes our results.

Corollary 1 (Revelation of customer flexibility data). *Consider an arbitrary location x on the turf of firm i. The quality of the additionally revealed customer information in the first period depends on how strongly consumers differ in their preferences and firm discount factor.*
i) If consumers are relatively homogeneous, $l \leq h_1 (\delta)$, no additional information is revealed in the first period independently of the discount factor, such that $\alpha^* (\delta) = 0$ for any δ.
ii) In all other cases firms obtain additional customer information on flexibility. How a higher discount factor influences its quality, depends on the intensity of consumer differentiation: If $l \lessapprox 2.64$, then $\partial \alpha^* (\delta) / \partial \delta > 0$ and the sign is opposite if $l \gtrapprox 2.67$. Finally, if $2.64 < l < 2.67$, then $\partial \alpha^* (\delta) / \partial \delta < 0$ when firms are relatively impatient and the sign is opposite otherwise.

Corollary 1 shows that the effect of a larger discount factor on the quality of the revealed customer data can be threefold. In particular, if l and/or δ are small so that consumers are *relatively homogeneous* ($l \leq h_1 (\delta)$), then this effect is absent and firms charge the same prices yielding the same market shares as if there were no second period. As explained above, this happens because the value of customer data is low when consumers are similar and/or when firms discount away future profits. When consumers are more differentiated and/or firms put sufficient weight on future profits, dynamic considerations matter for first-period prices and market shares. Whether in this case the distribution of consumers in the first period becomes more symmetric and, hence, customer data of a better quality is revealed, depends on how future profits respond to firms holding more precise data. As we showed in Proposition 1, the effect of better customer data (measured by α) on second-period profits tends to be positive when consumers are more similar in their preferences and negative otherwise. In the former case firms prefer more accurate customer data when the discount factor becomes larger, so that $\alpha^*(\delta)/\partial\delta > 0$ holds if $h_1(\delta) < l \lessapprox 2.64$ (or if $2.64 < l < 2.67$ and the discount factor is large). In the latter case firms prefer less precise information, so that $\alpha^*(\delta)/\partial\delta < 0$ if $l \gtrapprox 2.67$. However, in both cases firms acquire at least some customer data even if this reduces their second-period profits. This is because with sufficient heterogeneity in flexibility ($l > h_1(\delta)$) serving all consumers on a firm's turf in the first period would require setting excessively low prices.

An important result of Esteves (2010) is that firms may avoid learning consumer preferences to prevent tense competition in the subsequent period. In particular, she shows that the probability of the sharing outcome in the first period under which consumer types are fully revealed decreases when firms become more patient. We find an analogous result with $l \gtrapprox 2.67$, in which case less precise customer data is revealed when firms value future profits more. However, our model generates also the opposite result for $h_1(\delta) < l \lessapprox 2.64$, because more accurate behavior-based targeting is likely to increase profits in this case.

We conclude that by influencing the precision of revealed customer information in the first period, firms are able to strengthen the positive and dampen the negative effect of this information on second-period profits. We now turn to the question of how overall profits change compared to the case where firms are (for some exogenous reasons) not able to collect behavioral data and therefore can only discriminate along consumer locations in the second period. The following corollary summarizes our results, where we compare the discounted sum of profits in the subgame-perfect Nash equilibrium (in pure strategies) in both cases.

Corollary 2 (The profit effect of behavioral targeting). *The profit effect of combining mobile geo-targeting with behavior-based price discrimination is:*

i) neutral irrespective of the discount factor if $l \leq 1.5$,

ii) positive provided the discount factor is large enough and neutral otherwise if $1.5 < l < 2$,

iii) positive irrespective of the discount factor if $2 \leq l \lessapprox 3.07$,

iv) (weakly) positive if the discount factor is large enough and negative otherwise if $3.07 < l \lessapprox 4$, *v) negative irrespective of the discount factor if* $l > 4$.

Our results demonstrate that there are *pure* cases where the profit effect of targeted pricing based on consumer purchase histories depends only on their heterogeneity in preferences. Precisely, if $l \leq 1.5$, the ability of firms to engage in behavioral targeting is neutral for their discounted profits. When consumers do not differ a lot among each other, the value of additional customer data is small and no flexibility data is revealed in equilibrium making behavioral targeting irrelevant for profits. This result is in sharp contrast with Baye and Sapi (2017), where firms are strictly better off with additional customer data when consumers are quite homogeneous in their preferences. The reason is that in Baye and Sapi additional data is costless, while in our model firms have to sacrifice some of their first-period profits to gain it. When consumer differentiation is only modest, the value of this data to the firms is low, such that they prefer not to distort their optimal prices of the first period. As a result, Baye and Sapi overestimate the positive effect of additional data on profits.

If $2 \leq l \lessapprox 3.07$ ($l > 4$), the ability of firms to collect behavioral data is beneficial (detrimental) for their discounted profits. These results are consistent with the effect of flexibility data on second-period profits, as described in Proposition 1. Precisely, we showed there that profits are more likely to increase above the level without flexibility information if consumers are more homogeneous. In that case price competition is intensive even without behavioral data, so that additional customer data has mainly a positive rent-extraction effect as competition cannot increase much.

If the level of consumer differentiation takes intermediate values (not covered by the *pure* cases), then the sign of the profit effect of behavioral targeting is convoluted by the discount factor. Precisely, a higher weight on future profits makes this form of price discrimination profitable. This result is also driven by the effect of flexibility data on second-period profits as stated in Proposition 1. We showed there that when consumer differentiation is moderate, the effect of additional flexibility data on profits is related to the share $\alpha^* (\delta)$, which in turn depends on firm discount factor. This result is novel in the literature. Previous studies attributed unambiguous profit effects to price discrimination based on purchase histories independent of the discount factor (see Fudenberg and Tirole, 2000; Chen and Zhang, 2009; Esteves, 2010).[28]

[28] To make the results of Chen and Zhang (2009) comparable with ours, we need to set consumer discount factor to zero in their model. In this case firms are always better off with targeted pricing based on consumer purchase histories, irrespective of the firm discount factor (see Proposition 1).

We qualify these strict effects by allowing for different levels of consumer differentiation. This in turn influences the interplay between the rent-extraction and competition effects. When neither of these effects is strong enough, then the discount factor becomes the determining factor. We now turn to the analysis of how firms' ability to combine behavior-based price discrimination with geo-targeting influences consumer surplus and social welfare.[29]

Corollary 3 (The welfare effect of behavioral targeting). *The effect of combining mobile geo-targeting with behavior-based price discrimination on social welfare (consumer surplus) is:*
i) neutral irrespective of the discount factor if $l \leq 1.5$*,*
ii) negative if the discount factor is large enough and neutral otherwise if $1.5 < l < 2$*,*
iii) negative irrespective of the discount factor if $2 \leq l < 2.28$ *(*$2 \leq l < 2.61$*),*
iv) negative if the discount factor is large enough and positive otherwise if $2.28 \lessapprox l \lessapprox 2.67$ *(*$2.61 \lessapprox l \lessapprox 2.67$*),*
v) positive irrespective of the discount factor if $l > 2.67$*.*

Comparing the impact of the firm ability to engage in behavioral targeting on their profits and welfare, we conclude the following. If consumers are very similar in their preferences ($l \leq 1.5$), both firms serve all customers located closer to them in the first period and no flexibility data is revealed. As a result, both profits and welfare do not depend on whether firms can target consumers based on their behavior. When firms do gain flexibility data in equilibrium ($l > 1.5$), firms' and social welfare's interests are likely to be opposed. Additional customer data renders the second-period distribution of consumers more efficient, because more consumers buy from the firm located closer. This reduces transport costs and improves social welfare. However, with more homogeneous consumers (l is relatively small), firms distort first-period prices in order to obtain more flexibility data leading to a higher misalignment of consumers between the firms: The more flexible of them purchase from the firm located farther away. In this case firms benefit from behavioral data but social welfare reduces. This result is reversed when consumers become more differentiated (l is relatively large), because behavioral data in that case harms firms. They therefore consciously weaken information revelation in the first period making the distribution of consumers more efficient, because less customers buy from a farther firm. A similar pattern follows from the comparison of a change in profits and consumer surplus. From Corollaries 2 and

[29] To keep the exposition as simple as possible, we do not mention in the corollary a very special case of $2.28 \lessapprox l < 2.29$ ($2.61 \lessapprox l < 2.62$), where social welfare (consumer surplus) increases when the discount factor takes intermediate values and decreases otherwise.

3 we can also conclude that firm and consumer interests are not necessarily opposed. Precisely, if $2.67 < l \lessapprox 3.07$, then profits as well as consumer surplus (social welfare) increase by adding behavioral price discrimination to geo-targeting.

As in the case of the profit effect of behavior-based price discrimination, the way how the latter influences social welfare and consumer surplus also depends on firm discount factor when consumers differ moderately in their preferences. Precisely, consumers (and the overall welfare) are more likely to gain from firms combining behavioral pricing with mobile geo-targeting when the latter discount future profits more and are, hence, more likely to be worse off.

5 Conclusion

This paper analyzes a model taking into account four important features of a modern mobile targeting environment. First, sellers can observe consumers' real-time locations. Second, apart from location, there are other factors influencing the responsiveness of a consumer to discounts, such as age, income and occupation. Different from location, these are imperfectly observable by marketers. Third, sellers may infer consumer responsiveness (flexibility) from the observed previous purchasing behavior of a customer. Fourth, firms can deliver personalized offers through mobile devices in a private manner based on both consumer locations and their flexibility inferred from the previous purchase decisions. Our results show that firms benefit from the ability to collect behavioral data and use it for personalized pricing in mobile geo-targeting when consumers differ moderately in their preferences. With less differentiated consumers behavior-based price discrimination is neutral for profits, while with strongly differentiated consumers it intensifies competition and reduces profits. Different from the previous studies, our results also highlight the importance of the discount factor for the profit effect of behavioral targeting, which is likely to be positive when firms are more patient. We also find that consumer and firm interests are not necessarily opposed. In particular, when customers differ modestly in their preferences both consumer surplus and profits can increase with behavioral targeting leading to a higher social welfare. Finally, we show that firms strategically influence the quality of the (revealed) consumer behavioral data so as to enable higher rents extraction in case the data allows them to do so, and reduce the profit loss if data intensifies competition.

Our results are relevant for managers and policy alike. The main managerial implication of our results is that combining behavioral marketing with geo-targeting needs very careful consideration of the market environment. We highlight the role of consumer heterogeneity and firm discount factor and derive precise conditions under which such a campaign may be profitable in a competitive landscape. The main policy message relates to consumer and privacy policy: Combining behavioral price discrimination with geo-targeting can be both beneficial and harmful for consumers. While geo-targeting has been argued to typically foster competition (e.g., Thisse and Vives, 1988), combining it with behavioral price discrimination can turn around this effect, giving scope for a careful consumer policy. For example, restricting firms in their collection of types of data, such as age and demographics, that relate to their flexibility may improve consumer outcomes when these do not differ strongly among each other. Similarly, decreasing the data retention period (a proxy for the discount factor in our model) may also benefit consumers when these are moderately differentiated.

Appendix

Proof of Lemma 1. As firms are symmetric, we will restrict attention to the turf of firm A. Consider some $x < 1/2$ and segment α. Maximizing the expected profit of firm A yields the best-response function, which depends on the ratio $t^\alpha/\underline{t}$. If $t^\alpha/\underline{t} \leq 2$ $(\alpha \leq 1/(l-1))$, then $p_A^\alpha(x; p_B^\alpha) = p_B^\alpha + \underline{t}(1-2x)$, such that firm A optimally serves all consumers on segment α irrespective of firm B's price. Then in equilibrium firm B charges $p_B^\alpha(x) = 0$, because it would have an incentive to deviate from any positive price. Hence, $p_A^\alpha(x) = \underline{t}(1-2x)$. If $t^\alpha/\underline{t} > 2$, then the best response of firm A takes the form:

$$p_A^\alpha(x; p_B^\alpha) = \begin{cases} p_B^\alpha + \underline{t}(1-2x) & \text{if} \quad p_B^\alpha \geq (t^\alpha - 2\underline{t})(1-2x) \\ \frac{p_B^\alpha + t^\alpha(1-2x)}{2} & \text{if} \quad p_B^\alpha < (t^\alpha - 2\underline{t})(1-2x), \end{cases} \tag{2}$$

such that firm A serves all consumers on segment α only if the rival's price is relatively high. Maximization of the expected profit of firm B yields the best-response function:

$$p_B^\alpha(x; p_A^\alpha) = \begin{cases} \text{any } p_B^\alpha & \text{if} \quad p_A^\alpha \leq \underline{t}(1-2x) \\ \frac{p_A^\alpha - \underline{t}(1-2x)}{2} & \text{if} \quad \underline{t}(1-2x) < p_A^\alpha < (2t^\alpha - \underline{t})(1-2x) \\ p_A^\alpha - t^\alpha(1-2x) & \text{if} \quad p_A^\alpha \geq (2t^\alpha - \underline{t})(1-2x). \end{cases} \tag{3}$$

Inspecting (2), we conclude that firm B cannot serve all consumers in equilibrium. It is straightforward to show that there are no such prices, which constitute the equilibrium, where firm A serves all consumers. Hence, only the equilibrium can exist, where both firms serves consumers. Solving (2) and (3) simultaneously, we get the prices: $p_A^{\alpha}(x) = \underline{t}\,(1-2x)\,[2\alpha\,(l-1)+1]\,/3$ and $p_B^{\alpha}(x) = \underline{t}\,(1-2x)\,[\alpha\,(l-1)-1]\,/3$. For this equilibrium to exist, it must hold that $t^{\alpha}/\underline{t} > 2$. In a similar way one can derive the equilibrium on the segment $1-\alpha$. Precisely, if $1/t^{\alpha} \leq 2$ ($\alpha \geq (l-2)\,/\,[2\,(l-1)]$), then in the monopoly equilibrium firm A serves all consumers, where firms charge prices: $p_A^{1-\alpha}(x) = \underline{t}\,[1+\alpha\,(l-1)]\,(1-2x)$ and $p_B^{1-\alpha}(x) = 0$. If $1/t^{\alpha} > 2$, then the sharing equilibrium emerges with the prices: $p_A^{\alpha}(x) = \underline{t}\,(1-2x)\,[2l-1-\alpha\,(l-1)]\,/3$ and $p_B^{\alpha}(x) = \underline{t}\,(1-2x)\,[l-2-2\alpha\,(l-1)]\,/3$. *Q.E.D.*

Proof of Lemma 2. Lemma 2 follow directly from Lemma 1 given the following results: $1/\,(l-1) > (l-2)\,/\,[2\,(l-1)]$ if $l < 4$, $(l-2)\,/\,[2\,(l-1)] > 0$ if $l > 2$, $1/\,(l-1) > 1$ if $l < 2$, with the opposite sign otherwise. Note that $1/\,(l-1) > 0$ and $(l-2)\,/\,[2\,(l-1)] < 1$ hold for any l. *Q.E.D.*

Proof of Proposition 1. Consider first some x on the turf of firm A. We start with deriving firms' profits on each segment depending on α. Consider first segment α. If $\alpha \leq 1/\,(l-1)$, then firm A serves all consumers and profits are

$$\begin{aligned}\frac{\Pi_A^{\alpha}(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_A^{\alpha,1}(l;\alpha) := \frac{t^{\alpha}-\underline{t}}{1-\underline{t}} = \alpha \text{ and}\\ \frac{\Pi_B^{\alpha}(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_B^{\alpha,1}(l;\alpha) := 0.\end{aligned}$$

If $\alpha > 1/\,(l-1)$, then firm A serves consumers with $t \geq \underline{t}\,[\alpha\,(l-1)+2]\,/3$ and profits are

$$\begin{aligned}\frac{\Pi_A^{\alpha}(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_A^{\alpha,2}(l;\alpha) := \left[t^{\alpha} - \frac{\underline{t}(\alpha(l-1)+2)}{3}\right]\frac{[2\alpha(l-1)+1]}{3(1-\underline{t})} = \frac{[2\alpha(l-1)+1]^2}{9(l-1)} \text{ and}\\ \frac{\Pi_B^{\alpha}(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_B^{\alpha,2}(l;\alpha) := \left[\frac{\underline{t}(\alpha(l-1)+2)}{3} - \underline{t}\right]\frac{[\alpha(l-1)-1]}{3(1-\underline{t})} = \frac{[\alpha(l-1)-1]^2}{9(l-1)}.\end{aligned}$$

Consider now segment $1-\alpha$. If $\alpha \geq (l-2)\,/\,[2\,(l-1)]$, then firm A gains all consumers and firms realize profits:

$$\begin{aligned}\frac{\Pi_A^{1-\alpha}(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_A^{1-\alpha,1}(l;\alpha) := \frac{(1-t^{\alpha})t^{\alpha}}{(1-\underline{t})\underline{t}} = (1-\alpha)\,[1+\alpha\,(l-1)] \text{ and}\\ \frac{\Pi_B^{1-\alpha}(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_B^{1-\alpha,1}(l;\alpha) := 0.\end{aligned}$$

If $\alpha < (l-2)/[2(l-1)]$, then firm A serves consumers with $t \geq \underline{t}[l+1+\alpha(l-1)]/3$ and firms realize profits:

$$\begin{aligned}\frac{\Pi_A^{1-\alpha}(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_A^{1-\alpha,2}(l;\alpha) := \left[1 - \frac{\underline{t}[l+1+\alpha(l-1)]}{3}\right]\frac{[2l-1-\alpha(l-1)]}{3(1-\underline{t})} = \frac{[2l-1-\alpha(l-1)]^2}{9(l-1)} \text{ and} \\ \frac{\Pi_B^{1-\alpha}(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_B^{1-\alpha,2}(l;\alpha) := \left[\frac{\underline{t}[l+1+\alpha(l-1)]}{3} - t^\alpha\right]\frac{[l-2-2\alpha(l-1)]}{3(1-\underline{t})} = \frac{[l-2-2\alpha(l-1)]^2}{9(l-1)}.\end{aligned}$$

The profits on some x on the turf of firm B can be derived in a similar way. Note now that $\int_0^{1/2}(1-2x)\,dx = \int_{1/2}^1(2x-1)\,dx = 1/4$. Using the above results, we can write down the total profits depending on l and α under the assumption that on any x on its turf in the first period every firm served consumers with $t \geq t^\alpha$.

Consider first $l \leq 2$. The total profits of firm $i = A, B$ on both turfs are

$$\frac{4\Pi_i(l;\alpha)}{\underline{t}} = \Pi_A^{\alpha,1}(\cdot) + \Pi_A^{1-\alpha,1}(\cdot) = f_1(l;\alpha) := \alpha + (1-\alpha)[1+\alpha(l-1)].$$

Taking the derivative of $f_1(l;\alpha)$ with respect to α we get

$$\frac{\partial f_1(l;\alpha)}{\partial\alpha} = (1-2\alpha)(l-1),$$

such $f_1(l;\alpha)$ is given by the inverted U-shaped function of α, which gets its maximum at $\alpha = 1/2$.

Consider now $2 < l < 4$ and $\alpha \leq (l-2)/[2(l-1)]$, then the total profits of firm i on both turfs are

$$\frac{4\Pi_i(l;\alpha)}{\underline{t}} = \Pi_A^{\alpha,1}(\cdot) + \Pi_A^{1-\alpha,2}(\cdot) + \Pi_B^{1-\alpha,2}(\cdot) = f_2(l;\alpha) := \alpha + \frac{[2l-1-\alpha(l-1)]^2}{9(l-1)} + \frac{[l-2-2\alpha(l-1)]^2}{9(l-1)}.$$

Taking the derivative of $f_2(l;\alpha)$ with respect to α we get

$$\frac{\partial f_2(l;\alpha)}{\partial\alpha} = \frac{10\alpha(l-1)-8l+19}{9} > 0 \text{ if } \alpha > \alpha_2 := \frac{8l-19}{10(l-1)}.$$

Note that $\alpha_2 \leq 0$ if $l \leq 19/8 \approx 2.38$ and $\alpha_2 < (l-2)/[2(l-1)]$ if $l < 3$. Hence, if $2 < l \leq 19/8$, then $f_2(l;\alpha)$ increases in α. If $19/8 < l < 3$, then $f_2(l;\alpha)$ decreases till $\alpha = \alpha_2$ and increases afterwards. Finally, if $3 \leq l < 4$, then $f_2(l;\alpha)$ decreases in α.

If $(l-2)/[2(l-1)] < \alpha < 1/(l-1)$, then the total profits of firm i on both turfs are

$$\frac{4\Pi_i(l;\alpha)}{\underline{t}} = \Pi_A^{\alpha,1}(\cdot) + \Pi_A^{1-\alpha,1}(\cdot) = f_3(l;\alpha) := \alpha + (1-\alpha)[1+\alpha(l-1)].$$

Taking the derivative of $f_3(l;\alpha)$ with respect to α we get

$$\frac{\partial f_3(l;\alpha)}{\partial \alpha} = (1-2\alpha)(l-1) > 0 \text{ if } \alpha < \tfrac{1}{2}.$$

Note that $(l-2)/[2(l-1)] < 1/2$ for any l and $1/(l-1) < 1/2$ if $l > 3$. Hence, if $2 < l \leq 3$, then on $(l-2)/[2(l-1)] < \alpha < 1/(l-1)$, $f_3(l;\alpha)$ increases in α till $\alpha = 1/2$ and decreases afterwards. If $3 < l < 4$, then $f_3(l;\alpha)$ increases in α on $(l-2)/[2(l-1)] < \alpha < 1/(l-1)$.

If $\alpha \geq 1/(l-1)$, then the total profits of firm i on both turfs are

$$\begin{aligned}\frac{4\Pi_i(l;\alpha)}{\underline{t}} &= \Pi_A^{\alpha,2}(\cdot) + \Pi_B^{\alpha,2}(\cdot) + \Pi_A^{1-\alpha,1}(\cdot) \\ &= f_4(l;\alpha) := \frac{[2\alpha(l-1)+1]^2}{9(l-1)} + \frac{[\alpha(l-1)-1]^2}{9(l-1)} + (1-\alpha)[1+\alpha(l-1)].\end{aligned}$$

Taking the derivative of $f_4(l;\alpha)$ with respect to α we get

$$\frac{\partial f_4(l;\alpha)}{\partial \alpha} = \frac{9l-8\alpha(l-1)-16}{9} > 0 \text{ if } \alpha < \alpha_4 := \frac{9l-16}{8(l-1)}.$$

Note that $\alpha_4 < 1/(l-1)$ if $l < 24/9 \approx 2.67$ and $\alpha_4 < 1$ for any $2 < l < 4$. Hence, if $2 < l < 24/9$, then $f_4(l;\alpha)$ decreases in α. If $24/9 \leq l < 4$, then $f_4(l;\alpha)$ increases till $\alpha = \alpha_4$ and decreases afterwards.

Consider finally $l \geq 4$. If $\alpha \leq 1/(l-1)$, then the total profits of firm i on both turfs are

$$\begin{aligned}\frac{4\Pi_i(l;\alpha)}{\underline{t}} &= \Pi_A^{\alpha,1}(\cdot) + \Pi_A^{1-\alpha,2}(\cdot) + \Pi_B^{1-\alpha,2}(\cdot) \\ &= f_5(l;\alpha) := \alpha + \frac{[2l-1-\alpha(l-1)]^2}{9(l-1)} + \frac{[l-2-2\alpha(l-1)]^2}{9(l-1)}.\end{aligned}$$

Taking the derivative of $f_5(l;\alpha)$ with respect to α we get

$$\frac{\partial f_5(l;\alpha)}{\partial \alpha} = \frac{10\alpha(l-1)-8l+19}{9} > 0 \text{ if } \alpha > \alpha_5 := \frac{8l-19}{10(l-1)}.$$

Note that for any $l \geq 4$ it holds that $\alpha_5 > 1/(l-1)$. Hence, $f_5(l;\alpha)$ decreases in α.

If $1/(l-1) < \alpha < (l-2)/[2(l-1)]$, then the total profits of firm i on both turfs are

$$\begin{aligned}\tfrac{4\Pi_i(l;\alpha)}{\underline{t}} &= \Pi_A^{\alpha,2}(\cdot) + \Pi_B^{\alpha,2}(\cdot) + \Pi_A^{1-\alpha,2}(\cdot) + \Pi_B^{1-\alpha,2}(\cdot) \\ &= f_6(l;\alpha) := \tfrac{[2\alpha(l-1)+1]^2}{9(l-1)} + \tfrac{[\alpha(l-1)-1]^2}{9(l-1)} + \tfrac{[2l-1-\alpha(l-1)]^2}{9(l-1)} + \tfrac{[l-2-2\alpha(l-1)]^2}{9(l-1)}.\end{aligned}$$

Taking the derivative of $f_6(l;\alpha)$ with respect to α we get

$$\tfrac{\partial f_6(l;\alpha)}{\partial\alpha} = \tfrac{20\alpha(l-1)-4(2l-3)}{9} > 0 \text{ if } \alpha > \alpha_6 := \tfrac{2l-3}{5(l-1)}.$$

Note that for any $l \geq 4$ it holds that $1/(l-1) \leq \alpha_6 \leq (l-2)/[2(l-1)]$. Hence, $f_6(l;\alpha)$ decreases till $\alpha = \alpha_6$ and increases afterwards.

If $\alpha \geq (l-2)/[2(l-1)]$, then the total profits of firm i on both turfs are

$$\begin{aligned}\tfrac{4\Pi_i(l;\alpha)}{\underline{t}} &= \Pi_A^{\alpha,2}(\cdot) + \Pi_B^{\alpha,2}(\cdot) + \Pi_A^{1-\alpha,1}(\cdot) \\ &= f_7(l;\alpha) := \tfrac{[2\alpha(l-1)+1]^2}{9(l-1)} + \tfrac{[\alpha(l-1)-1]^2}{9(l-1)} + (1-\alpha)[1+\alpha(l-1)].\end{aligned}$$

Taking the derivative of $f_7(l;\alpha)$ with respect to α we get

$$\tfrac{\partial f_7(l;\alpha)}{\partial\alpha} = \tfrac{9l-16-8\alpha(l-1)}{9} > 0 \text{ if } \alpha < \alpha_7 := \tfrac{9l-16}{8(l-1)}.$$

Note that for any $l \geq 4$ it holds that $\alpha_7 > (l-2)/[2(l-1)]$. Moreover, $\alpha_7 > 1$ if $l > 8$, with an opposite inequality otherwise. Hence, if $4 \leq l \leq 8$, then $f_7(l;\alpha)$ increases till $\alpha = \alpha_7$ and decreases afterwards. If $l > 8$, then $f_7(l;\alpha)$ increases in α.

We can now summarize the results on the behavior of the total profits in α depending on l. *i)* If $l \leq 2.38$, then total profits are given by the inverted U-shaped function of α, which gets its maximum at $\alpha = 1/2$.

ii) If $2.38 < l < 2.67$, then total profits are given by a non-monotonic function, which gets a (local) minimum at $\alpha = (8l-19)/(10l-10)$ and a (local) maximum at $\alpha = 1/2$. This function decreases on the intervals: $[0, (8l-19)/(10l-10)]$ and $[1/2, 1]$, and increases on the remaining intervals. Note that

$$\tfrac{4\Pi_i\left(l;\frac{1}{2}\right)}{\underline{t}} = f_3\left(l;\tfrac{1}{2}\right) = \tfrac{l+3}{4} > \tfrac{4\Pi_i(l;0)}{\underline{t}} = f_2(l;0) = \tfrac{5l^2-8l+5}{9(l-1)}, \text{ for any } 2.38 < l < 2.67,$$

such that $\Pi_i(l;k)$ gets a global maximum at $\alpha = 1/2$. From this and the fact that $\Pi_i(l;k)$ is a continuous function of α, we conclude that there exists $(8l-19)/(10l-10) < \widehat{\alpha}(l) < 1/2$, such that $\Pi_i(l;\alpha) \geq \Pi_i(l;0)$ if $\alpha \geq \widehat{\alpha}(l)$, with an opposite inequality otherwise. $\widehat{\alpha}(l)$ is implicitly given either by the equation:

$$f_2(l;0) = \tfrac{5l^2-8l+5}{9(l-1)} = f_2(l;\widehat{\alpha}(l)) = \widehat{\alpha} + \tfrac{[2l-1-\widehat{\alpha}(l-1)]^2}{9(l-1)} + \tfrac{[l-2-2\widehat{\alpha}(l-1)]^2}{9(l-1)}, \tag{4}$$

or by the equation:

$$f_2(l;0) = \tfrac{5l^2-8l+5}{9(l-1)} = f_3(l;\widehat{\alpha}(l)) = \widehat{\alpha} + (1-\widehat{\alpha})[1+\widehat{\alpha}(l-1)]. \tag{5}$$

In the former case we get that

$$\tfrac{\partial\widehat{\alpha}(l)}{\partial l} = \tfrac{\alpha(5\alpha-8)}{8l-19-10\alpha(l-1)} > 0$$

and in the latter case we get

$$\tfrac{\partial\widehat{\alpha}(l)}{\partial l} = \tfrac{\alpha^2\left(9l^2-18l+9\right)+\alpha\left(-9l^2+18l-9\right)+5l^2-10l+3}{9(1-2\alpha)(l-1)^3} > 0,$$

because if $2.38 < l < 2.67$, then $\alpha^2\left(9l^2-18l+9\right)+\alpha\left(-9l^2+18l-9\right)+5l^2-10l+3 > 0$ for any α.

iii) If $2.67 < l < 3$, then the function $\Pi_i(l;k)$ decreases on: $[0,(8l-19)/(10l-10)]$, $[1/2, 1/(l-1)]$ and $[(9l-16)/(8l-8), 1]$. The comparisons show that

$$\begin{aligned}
\tfrac{4\Pi_i\left(l;\frac{1}{2}\right)}{\underline{t}} &= f_3\left(l;\tfrac{1}{2}\right) = \tfrac{l+3}{4} \geq \tfrac{4\Pi_i\left(l;\frac{9l-16}{8(l-1)}\right)}{\underline{t}} = f_4\left(l;\tfrac{9l-16}{8(l-1)}\right) = \tfrac{9l^2-16l+16}{16(l-1)} \text{ if } 2.67 < l \leq 2.8,\\
\tfrac{4\Pi_i\left(l;\frac{1}{l-1}\right)}{\underline{t}} &= f_3\left(l;\tfrac{1}{l-1}\right) = f_4\left(l;\tfrac{1}{l-1}\right) = \tfrac{2l-3}{l-1} > f_2(l;0) = f_4(l;1) = \tfrac{5l^2-8l+5}{9(l-1)} \text{ if } 2.67 < l < 3.
\end{aligned}$$

We make two conclusions. First, $\Pi_i(l;\alpha)$ gets the global maximum at $\alpha = 1/2$ if $l \leq 2.8$ and at $\alpha = (9l-16)/[8(l-1)]$ otherwise. Second, using the fact that $\Pi_i(l;\alpha)$ is a continuos function of α, we conclude that there exists $(8l-19)/(10l-10) < \widehat{\alpha}(l) < 1/2$, such that $\Pi_i(l;\alpha) \geq \Pi_i(l;0)$ if $\alpha \geq \widehat{\alpha}(l)$, with an opposite inequality otherwise. As in the previous case, $\widehat{\alpha}(l)$ is given by either (4) or (5). As we showed above, in both cases $\partial\widehat{\alpha}(l)/\partial l > 0$ holds.

iv) If $3 < l \leq 4$, then $\Pi_i(l;\alpha)$ decreases on: $[0,(l-2)/(2l-2)]$ and $[(9l-16)/(8l-8), 1]$, while increases on the remaining interval. Note that

$$f_2(l;0) = f_4(l;1) = \tfrac{5l^2-8l+5}{9(l-1)} < f_4\left(l;\tfrac{9l-16}{8(l-1)}\right) = \tfrac{9l^2-16l+16}{16(l-1)} \text{ for any } l, \tag{6}$$

such that $\Pi_i(l;\alpha)$ has a global maximum at $\alpha = (9l-16)/(8l-8)$. As $\Pi_i(l;\alpha)$ is a continuous function of α, we conclude that there exists $(l-2)/(2l-2) < \widehat{\alpha}(l) < (9l-16)/(8l-8)$, such that $\Pi_i(l;\alpha) \geq \Pi_i(l;0)$ if $\alpha \geq \widehat{\alpha}(l)$, with an opposite inequality otherwise. As in the previous case, $\widehat{\alpha}(l)$ is given by either (4) or (5). As we showed above, in both cases $\partial\widehat{\alpha}(l)/\partial l > 0$ holds.

v) If $4 < l < 8$, then $\Pi_i(l;\alpha)$ decreases on: $[0,(2l-3)/(5l-5)]$ and $[(9l-16)/(8l-8),1]$, while increases on the remaining interval. Due to (6), $\Pi_i(l;\alpha)$ has a global maximum at $\alpha = (9l-16)/(8l-8)$. As $\Pi_i(l;\alpha)$ is a continuous function of α, we conclude that there exists $(2l-3)/(5l-5) < \widehat{\alpha}(l) < (9l-16)/(8l-8)$, such that $\Pi_i(l;\alpha) \geq \Pi_i(l;0)$ if $\alpha \geq \widehat{\alpha}(l)$, with an opposite inequality otherwise. $\widehat{\alpha}(l)$ is implicitly given either by the equation:

$$f_5(l;0) = \tfrac{5l^2-8l+5}{9(l-1)} = f_6(l;\widehat{\alpha}(l)) = \tfrac{[2\widehat{\alpha}(l-1)+1]^2}{9(l-1)} + \tfrac{[\widehat{\alpha}(l-1)-1]^2}{9(l-1)} + \tfrac{[2l-1-\widehat{\alpha}(l-1)]^2}{9(l-1)} + \tfrac{[l-2-2\widehat{\alpha}(l-1)]^2}{9(l-1)},$$

or by the equation:

$$f_5(l;0) = \tfrac{5l^2-8l+5}{9(l-1)} = f_7(l;\widehat{\alpha}(l)) = \tfrac{[2\widehat{\alpha}(l-1)+1]^2}{9(l-1)} + \tfrac{[\widehat{\alpha}(l-1)-1]^2}{9(l-1)} + (1-\widehat{\alpha})[1+\widehat{\alpha}(l-1)].$$

In the former case we get that

$$\begin{aligned}\tfrac{\partial\widehat{\alpha}(l)}{\partial l} &= -\tfrac{(-5l^2+10l-5)[\alpha-\alpha_1(l)][\alpha-\alpha_2(l)]}{2(l-1)^2[2l-3-\alpha(5l-5)]}, \text{ where}\\ \alpha_1(l) &= \tfrac{-(4l^2-8l+4)+2(l-1)\sqrt{4l^2-8l+9}}{2(-5l^2+10l-5)} \text{ and } \alpha_2(l) = \tfrac{-(4l^2-8l+4)-2(l-1)\sqrt{4l^2-8l+9}}{2(-5l^2+10l-5)}.\end{aligned}$$

Note that as $f_6(l;\alpha)$ is defined on $1/(l-1) < \alpha < (l-2)/[2(l-1)]$, while $\alpha_1(l) < 1/(l-1)$ and $\alpha_2(l) > (l-2)/[2(l-1)]$ for any $4 < l < 8$, then $\alpha - \alpha_1(l) > 0$ and $\alpha - \alpha_2(l) < 0$. Finally, as $-5l^2 + 10l - 5 < 0$ for any $4 < l < 8$ and $\widehat{\alpha}(l) > (2l-3)/(5l-5)$, we conclude that $\partial\widehat{\alpha}(l)/\partial l > 0$. In the latter case we get

$$\tfrac{\partial\widehat{\alpha}(l)}{\partial l} = -\tfrac{4(\alpha-1)\left(\alpha-\frac{5}{4}\right)}{8\alpha(l-1)-(9l-16)} > 0 \text{ as } \widehat{\alpha}(l) < \tfrac{9l-16}{8\alpha(l-1)}.$$

vi) If $l \geq 8$, then $\Pi_i(l;\alpha)$ is a U–shaped function, which gets its (global) minimum at $\alpha =$

$(2l-3)/(5l-5)$. *Q.E.D.*

Proof of Proposition 2. Consider some x on the turf of firm A. Using the results of Lemma 2 and the notation from the proof of Proposition 1, we can write down second-period profits at x depending on l and α. If $l \leq 2$, then firm A gains all consumers at x independently of α, such that profits of firm $i = A, B$ at x, $\Pi_i\left(x \middle| x < 1/2\right)$, are

$$\begin{aligned} \tfrac{\Pi_A(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_A^{\alpha,1}(l;\alpha) + \Pi_A^{1-\alpha,1}(l;\alpha) = \alpha + (1-\alpha)\left[1+\alpha(l-1)\right], \\ \tfrac{\Pi_B(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_B^{\alpha,1}(l;\alpha) + \Pi_B^{1-\alpha,1}(l;\alpha) = 0. \end{aligned} \tag{7}$$

Consider now $2 < l < 4$, in which case second-period profits at x depend on α. If $\alpha \leq (l-2)/[2(l-1)]$, then firm A gains all consumers on α and loses some consumers on $1-\alpha$, such that profits are

$$\begin{aligned} \tfrac{\Pi_A(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_A^{\alpha,1}(l;\alpha) + \Pi_A^{1-\alpha,2}(l;\alpha) = \alpha + \tfrac{[2l-1-\alpha(l-1)]^2}{9(l-1)}, \\ \tfrac{\Pi_B(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_B^{\alpha,1}(l;\alpha) + \Pi_B^{1-\alpha,2}(l;\alpha) = \tfrac{[l-2-2\alpha(l-1)]^2}{9(l-1)}. \end{aligned} \tag{8}$$

If $(l-2)/[2(l-1)] \leq \alpha < 1/(l-1)$, then firm A serves all consumers on both segments, and profits are given by (7). If $\alpha \geq 1/(l-1)$, then firm A loses consumers on α, and profits are

$$\begin{aligned} \tfrac{\Pi_A(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_A^{\alpha,2}(l;\alpha) + \Pi_A^{1-\alpha,1}(l;\alpha) = \tfrac{[2\alpha(l-1)+1]^2}{9(l-1)} + (1-\alpha)\left[1+\alpha(l-1)\right], \\ \tfrac{\Pi_B(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_B^{\alpha,2}(l;\alpha) + \Pi_B^{1-\alpha,1}(l;\alpha) = \tfrac{[\alpha(l-1)-1]^2}{9(l-1)}. \end{aligned} \tag{9}$$

Consider finally $l \geq 4$. If $\alpha \leq 1/(l-1)$, then firm A loses consumers on $1-\alpha$, and profits are given by (8). If $1/(l-1) < \alpha < (l-2)/[2(l-1)]$, then firm A loses consumers on both segments, and firms realize profits:

$$\begin{aligned} \tfrac{\Pi_A(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_A^{\alpha,2}(l;\alpha) + \Pi_A^{1-\alpha,2}(l;\alpha) = \tfrac{[2\alpha(l-1)+1]^2}{9(l-1)} + \tfrac{[2l-1-\alpha(l-1)]^2}{9(l-1)}, \\ \tfrac{\Pi_B(x|x<1/2)}{\underline{t}(1-2x)} &= \Pi_B^{\alpha,2}(l;\alpha) + \Pi_B^{1-\alpha,2}(l;\alpha) = \tfrac{[\alpha(l-1)-1]^2}{9(l-1)} + \tfrac{[l-2-2\alpha(l-1)]^2}{9(l-1)}. \end{aligned} \tag{10}$$

If $\alpha \geq (l-2)/[2(l-1)]$, then firm A loses consumers on α, and profits are given by (9).

We introduce now a new notation for the (adjusted) price of firm $i = A, B$ on some $x < 1/2$:

$$p_i^x := \frac{p_A(x)}{(1-2x)\underline{t}}.$$

At any $x < 1/2$ those consumers buy at firm A who have relatively high transport costs:

$$\begin{aligned} t &\geq t^{\alpha}\left(p_A\left(x\right), p_B\left(x\right)\right) = \alpha + \underline{t}\left(1-\alpha\right), \text{ where} \\ t^{\alpha}\left(\cdot\right) &= \frac{p_A(x)-p_B(x)}{1-2x} = \left(p_A^x - p_B^x\right)\underline{t}, \end{aligned} \tag{11}$$

from where we can derive α as follows

$$\alpha = \frac{p_A(x)-p_B(x)}{(1-2x)\underline{t}(l-1)} - \frac{1}{l-1} = \frac{p_A^x - p_B^x - 1}{l-1}. \tag{12}$$

Note next that if $\underline{t} \leq t^{\alpha}\left(\cdot\right) \leq 1$, then the profit of firm A at $x < 1/2$ in the first period is

$$\begin{aligned} \left[1 - \frac{p_A(x)-p_B(x)}{1-2x}\right]\frac{p_A(x)}{1-\underline{t}} &= \underline{t}\left(1-2x\right)\left[l - \frac{p_A(x)-p_B(x)}{(1-2x)\underline{t}}\right]\frac{p_A(x)}{\underline{t}(1-2x)(l-1)} \\ &= \frac{\underline{t}(1-2x)\left(l-p_A^x+p_B^x\right)p_A^x}{l-1}. \end{aligned}$$

Similarly, the profit of firm B at $x < 1/2$ in the first period is

$$\left[\frac{p_A(x)-p_B(x)}{1-2x} - \underline{t}\right]\frac{p_B(x)}{1-\underline{t}} = \frac{\underline{t}(1-2x)\left(p_A^x-p_B^x-1\right)p_B^x}{(l-1)}.$$

To derive the optimal prices of the first period, we will consider the discounted sum of each firm's profits over two periods, multiplied by $(l-1)$ and divided by $\underline{t}\,(1-2x)$.

Part 1. Consider first $l \leq 2$, in which case firm A chooses p_A^x to maximize the profits:

$$\begin{aligned} & \left(l - p_A^x + p_B^x\right)p_A^x + \delta\left(l-1\right)\left[\alpha + \left(1-\alpha\right)\left(1+\alpha\left(l-1\right)\right)\right] \\ = & \left(l - p_A^x + p_B^x\right)p_A^x + \delta\left[p_A^x - p_B^x - 1 + \left(l - p_A^x + p_B^x\right)\left(p_A^x - p_B^x\right)\right]. \end{aligned} \tag{13}$$

Firm B chooses p_B^x to maximize the profits:

$$\left(p_A^x - p_B^x - 1\right)p_B^x. \tag{14}$$

Solving firms' first-order conditions we arrive at the prices:

$$p_A^{x*} = \frac{2l(1+\delta)-1}{2\delta+3} \text{ and } p_B^{x*} = \frac{l-2-\delta(1-l)}{2\delta+3}. \tag{15}$$

Note that second-order conditions are also fulfilled. For the prices (15) to constitute the equilibrium, it must hold that $\underline{t} < t^{\alpha}\left((1-2x)\,\underline{t}p_A^{x*}, (1-2x)\,\underline{t}p_B^{x*}\right) \leq 1$, which yields the condition:

$$\frac{1}{1+l} < \frac{1+\delta}{3+2\delta} \leq \frac{l}{1+l}. \tag{16}$$

Note that $l/(1+l) > 0.5$ for any l and $(1+\delta)/(3+2\delta) \leq 0.4$ for any δ, such that the right-hand side of (16) is fulfilled for any δ and any l. The left-hand side of (16) is fulfilled if

$$l > \bar{l}_1(\delta) := \frac{2+\delta}{1+\delta}. \tag{17}$$

It holds that $1.5 \leq \bar{l}_1(\delta) \leq 2$ for any δ and $\partial\bar{l}(\delta)/\partial\delta < 0$. Note finally that if (17) holds, then $p_A^{x*} > 0$ and $p_B^{x*} > 0$.

If $l \leq \bar{l}_1(\delta)$, then the monopoly equilibrium emerges, where firm A serves all consumers at x. In this equilibrium firm A charges the highest price at which it can gain all consumers:

$$p_A(x, p_B(x)) = p_B(x) + \underline{t}(1-2x). \tag{18}$$

It follows from (18) that $p_B^*(x) = 0$, because firm B would have an incentive to deviate downwards from any positive price. Hence,

$$p_A^{x*}(x) = 1 \text{ and } p_B^{x*}(x) = 0. \tag{19}$$

For the prices (19) to constitute the equilibrium, none of the firms should have an incentive to deviate. Precisely, firm A should not have an incentive to increase its price: the derivative of (13) evaluated at $p_A^x = 1$ and $p_B^x = 0$ must be non-positive, which yields the condition $l \leq \bar{l}_1(\delta)$, which is the opposite of (17).

Part 2. Consider now $2 \leq l \leq 4$, in which case second-period profits are given by different functions depending on α.

i) Consider first $\alpha \leq (l-2)/[2(l-1)]$. The profits of the second period are then given by

(8). Firm A chooses p_A^x to maximize the profits:

$$\begin{aligned} & \left(l - p_A^x + p_B^x\right) p_A^x + \tfrac{\delta}{9}\left[9\alpha\left(l-1\right) + \left(2l - 1 - \alpha\left(l-1\right)\right)^2\right] \\ = \; & \left(l - p_A^x + p_B^x\right) p_A^x + \tfrac{\delta}{9}\left[9\left(p_A^x - p_B^x - 1\right) + \left(2l - p_A^x + p_B^x\right)^2\right]. \end{aligned} \tag{20}$$

Firm B chooses p_B^x to maximize the profits:

$$\left(p_A^x - p_B^x - 1\right) p_B^x + \tfrac{\delta\left[l-2-2\alpha(l-1)\right]^2}{9} = \left(p_A^x - p_B^x - 1\right) p_B^x + \tfrac{\delta\left(l - 2p_A^x + 2p_B^x\right)^2}{9}. \tag{21}$$

We first show that no monopoly equilibrium (where firm A serves all consumers at x) in the first period exists. Assume that there is such an equilibrium. In this equilibrium firm A will charge the price $p_A^x = p_B^x + 1$. Indeed, at a higher price firm A does not serve all consumers at x and at any lower price firm A realizes lower profits (first-period profits decrease, while second-period profits do not change). Firm B does not have an incentive to increase its price, because both the first-period and second-period profits do not change. However, one has to exclude the incentive of firm B to decrease its price. Similarly, one has to exclude the incentive of firm A to increase its price. Taking the derivatives of (20) and (21) with respect to p_A^x and p_B^x and evaluating them at $p_A^x = p_B^x + 1$, yields the following inequalities, respectively:

$$l + \tfrac{11}{9}\delta - \tfrac{4}{9}l\delta - 2 \leq p_B^x \text{ and } \tfrac{4}{9}l\delta - \tfrac{8}{9}\delta \geq p_B^x. \tag{22}$$

There exists p_B^x, which satisfies both inequalities in (22), only if

$$l \leq \tfrac{18-19\delta}{9-8\delta}. \tag{23}$$

Note that for any δ and any $l \geq 2$ (except for $\delta = 0$ and $l = 2$, which is covered by $l \leq \bar{l}_1(\delta)$ in *Part 1*), (23) does not hold, which proves that in the first period only the sharing equilibrium can exist.

In the sharing equilibrium first-order conditions have to be fulfilled. Solving them simultaneously, we arrive at the prices:

$$\begin{aligned} p_A^{x*} &= \tfrac{54l+60\delta-36l\delta-24\delta^2+8l\delta^2-27}{81-30\delta}, \\ p_B^{x*} &= \tfrac{27l+33\delta-12l\delta-24\delta^2+8l\delta^2-54}{81-30\delta}, \text{ which yield} \end{aligned} \tag{24}$$

$$\alpha\left(p_A^{x*},p_B^{x*}\right)=\tfrac{9l+19\delta-8l\delta-18}{(l-1)(27-10\delta)}. \tag{25}$$

Note that second-order conditions are fulfilled, and for any δ and any $l\geq 2$ it holds that $\alpha\left(p_A^{x*},p_B^{x*}\right)\geq 0$. Imposing $\alpha\left(p_A^{x*},p_B^{x*}\right)\leq (l-2)/\left[2\left(l-1\right)\right]$, we arrive at

$$l\geq \bar{l}_2\left(\delta\right):=\tfrac{6(1+\delta)}{3+2\delta}.$$

Note that $\partial\bar{l}_2\left(\delta\right)/\partial\delta>0$, $\bar{l}_2\left(0\right)=2$ and $\bar{l}_2\left(1\right)=2.4$. Note that $p_A^{x*}\geq 0$ stated in (24) requires

$$l\geq f_1(\delta):=\tfrac{24\delta^2-60\delta+27}{2\left(4\delta^2-18\delta+27\right)},$$

which is true for any δ and any l, because for any δ it holds that $f_1(\delta)<1$. Finally, $p_B^{x*}\geq 0$ stated in (24) requires

$$l\geq f_2(\delta):=\tfrac{24\delta^2-33\delta+54}{8\delta^2-12\delta+27},$$

which is true for any δ and any $l\geq 2$, because for any δ it holds that $f_2(\delta)\leq 2$.

ii) Consider now $(l-2)/\left[2\left(l-1\right)\right]\leq\alpha<1/\left(l-1\right)$. In this case the (adjusted) profits over two periods are given by (13) and (14), which yields the equilibrium prices (15) and the share of firm B in the first period:

$$\alpha\left(p_A^{x*},p_B^{x*}\right)=\tfrac{p_A^{x*}-p_B^{x*}-1}{l-1}=\tfrac{l(1+\delta)-(2+\delta)}{(3+2\delta)(l-1)}.$$

The condition $\alpha\left(p_A^{x*},p_B^{x*}\right)<1/\left(l-1\right)$ requires $l<\left(5+3\delta\right)/\left(1+\delta\right)$, which is fulfilled for any δ, because $\left(5+3\delta\right)/\left(1+\delta\right)\geq 4$ for any δ. The condition $\alpha\left(p_A^{x*},p_B^{x*}\right)\geq(l-2)/\left[2\left(l-1\right)\right]$ requires $l\leq\bar{l}_3\left(\delta\right):=2\left(1+\delta\right)$. Note that $\partial\bar{l}_3\left(\delta\right)/\partial\delta>0$, $\bar{l}_3\left(0\right)=2$ and $\bar{l}_3\left(1\right)=4$. Note finally that for any $2\leq l\leq 4$ and any δ, it holds that $p_A^{x*}>0$ and $p_B^{x*}\geq 0$.

iii) Consider finally $\alpha\geq 1/\left(l-1\right)$, in which case the profits of the second period are given by (9). Then firm A chooses p_A^x to maximize the profits:

$$\begin{aligned} & \left(l-p_A^x+p_B^x\right)p_A^x+\tfrac{\delta[2\alpha(l-1)+1]^2}{9}+\delta\left(1-\alpha\right)\left(l-1\right)\left[1+\alpha\left(l-1\right)\right] \\ = & \left(l-p_A^x+p_B^x\right)p_A^x+\tfrac{\delta\left(2p_A^x-2p_B^x-1\right)^2}{9}+\delta\left(l-p_A^x+p_B^x\right)\left(p_A^x-p_B^x\right). \end{aligned} \tag{26}$$

Firm B chooses p_B^x to maximize the profits:

$$\begin{aligned} & \left(p_A^x - p_B^x - 1\right) p_B^x + \tfrac{\delta[\alpha(l-1)-1]^2}{9} \\ = & \left(p_A^x - p_B^x - 1\right) p_B^x + \tfrac{\delta\left(p_A^x - p_B^x - 2\right)^2}{9}. \end{aligned} \quad (27)$$

Solving simultaneously first-order conditions of the firms, we arrive at the prices:

$$\begin{aligned} p_A^{x*} &= -\tfrac{-54l+42\delta-48l\delta-16\delta^2+6l\delta^2+27}{24\delta+81}, \\ p_B^{x*} &= -\tfrac{-27l+18\delta-21l\delta-16\delta^2+6l\delta^2+54}{24\delta+81}, \end{aligned} \quad (28)$$

which yield the share of firm B in the first period:

$$\alpha\left(p_A^{x*}, p_B^{x*}\right) = \tfrac{p_A^{x*} - p_B^{x*} - 1}{l-1} = \tfrac{9l(1+\delta)-16\delta-18}{(27+8\delta)(l-1)}. \quad (29)$$

Note that for any $l \geq 2$ and δ, $p_A^{x*} > 0$ and $p_B^{x*} \geq 0$ hold. The condition $\alpha(\cdot) \geq 1/(l-1)$ requires that

$$l \geq \bar{l}_4(\delta) := \tfrac{15+8\delta}{3(1+\delta)}.$$

Note that $\partial \bar{l}_4(\delta)/\partial\delta < 0$ and $\bar{l}_4(\delta) \leq 4$ if $\delta \geq 0.75$. The condition $\alpha(\cdot) \leq 1$ requires that

$$l \geq \tfrac{9-8\delta}{18-\delta}. \quad (30)$$

Since the the right-hand side of (30) is for any δ smaller than 1, then $\alpha(\cdot) \leq 1$ holds for any δ and any $l \geq 2$. Finally, it can be checked that second-order conditions are fulfilled.

Combining the results from the analysis of the cases *i)*, *ii)* and *iii)*, we conclude that depending on l and δ we have either one, two or three candidate equilibria. At the next step we have to find the equilibrium for any l and δ.

1) Consider first $\bar{l}_2(\delta) \leq l \leq \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$, which yield two candidate equilibria, (15) and (24).

1.a) Consider first the candidate equilibrium (24). It is straightforward that none of the firms has an incentive to deviate on $\alpha(\cdot) \leq (l-2)/[2(l-1)]$. We consider two other deviations by each of the firms.

Deviation on $(l-2)/[2(l-1)] \leq \alpha(\cdot) \leq 1/(l-1)$**.**

i) The incentives of firm A. If firm A deviates, then it realizes the profits (13). Maximizing (13) with respect to p_A^x and keeping p_B^x at p_B^{x*} given in (24), yields the deviation price:

$$p_A^{x,dev}\left(p_B^{x*}\right)=\frac{108l+6\delta+93l\delta+12\delta^2-48\delta^3-46l\delta^2+16l\delta^3-54}{-60\delta^2+102\delta+162}, \tag{31}$$

and the market share of firm B in the first period:

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right),p_B^{x*}\right)=-\frac{54\delta-54l-63l\delta-54\delta^2+38l\delta^2+108}{6(l-1)(27-10\delta)(1+\delta)}.$$

Note that $p_A^{x,dev}\left(p_B^{x*}\right)>0$ for any δ and any l. Imposing the requirement that

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right),p_B^{x*}\right)\geq\frac{l-2}{2(l-1)}, \tag{32}$$

we arrive at the constraint:

$$l\leq\frac{48\delta-6\delta^2+54}{8\delta^2-12\delta+27}. \tag{33}$$

Note that for any δ it holds that

$$\frac{48\delta-6\delta^2+54}{-12\delta+8\delta^2+27}>\min\left\{\bar{l}_3\left(\delta\right),\bar{l}_4\left(\delta\right)\right\},$$

such that for any l with $\bar{l}_2\left(\delta\right)\leq l\leq\min\left\{\bar{l}_3\left(\delta\right),\bar{l}_4\left(\delta\right)\right\}$, (33) holds and, hence, (32) is fulfilled. Imposing the requirement

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right),p_B^{x*}\right)\leq\frac{1}{l-1} \tag{34}$$

we arrive at the constraint:

$$l\leq\frac{156\delta-114\delta^2+270}{63\delta-38\delta^2+54}. \tag{35}$$

Note that for any δ it holds that

$$\frac{156\delta-114\delta^2+270}{63\delta-38\delta^2+54}>\min\left\{\bar{l}_3\left(\delta\right),\bar{l}_4\left(\delta\right)\right\},$$

such that for any l with $\bar{l}_2\left(\delta\right)\leq l\leq\min\left\{\bar{l}_3\left(\delta\right),\bar{l}_4\left(\delta\right)\right\}$, (35) holds and, hence, (34) is fulfilled. It follows that the price (31) is indeed the optimal deviation price of firm A on $\left(l-2\right)/\left[2\left(l-1\right)\right]\leq$

$\alpha(\cdot) \leq 1/(l-1)$. The difference between firm A's equilibrium and deviation profits is

$$f_1(l,\delta) := \frac{\delta\left[l^2\left(-100\delta^3+372\delta^2+531\delta+2268\right)-l\left(-120\delta^3+276\delta^2+6552\delta+6156\right)-360\delta^3+2520\delta^2+6120\delta+3240\right]}{36(\delta+1)(10\delta-27)^2}.$$

The function $f_1(l,\delta)$ opens upwards for any δ and has two roots:

$$\begin{aligned} l_1(\delta) &:= \frac{-120\delta^3+276\delta^2+6552\delta+6156-36(27-10\delta)(1+\delta)\sqrt{(\delta+1)(9-\delta)}}{2\left(-100\delta^3+372\delta^2+531\delta+2268\right)} \text{ and} \\ l_2(\delta) &:= \frac{-120\delta^3+276\delta^2+6552\delta+6156+36(27-10\delta)(1+\delta)\sqrt{(\delta+1)(9-\delta)}}{2\left(-100\delta^3+372\delta^2+531\delta+2268\right)}. \end{aligned} \tag{36}$$

For any δ it holds: $l_1(\delta) < \bar{l}_2(\delta)$ and $\bar{l}_2(\delta) \leq l_2(\delta) \leq \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$. Hence, for any l with $\bar{l}_2(\delta) \leq l \leq \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$ we have that $f_1(l,\delta) \geq 0$ if $l \geq l_2(\delta)$, while $f_1(l,\delta) < 0$ if $l < l_2(\delta)$. In the former case firm A does not have an incentive to deviate from p_A^{x*} in (24) and does in the latter.

ii) The incentives of firm B. If firm B deviates, then its profits are given by (14). Maximizing (14) with respect to p_B^x and keeping p_A^x at p_A^{x*} in (24), yields the deviation price:

$$p_B^{x,dev}\left(p_A^{x*}\right) = \frac{27l+45\delta-18l\delta-12\delta^2+4l\delta^2-54}{81-30\delta},$$

and the market share of firm B in the first period:

$$\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) = \frac{27l+45\delta-18l\delta-12\delta^2+4l\delta^2-54}{(l-1)(81-30\delta)}.$$

Note that for any δ and any $l \geq 2$ it holds that $p_B^{x,dev}\left(p_A^{x*}\right) \geq 0$. The comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) - \frac{l-2}{2(l-1)} = \frac{(9-4\delta)(6\delta-3l-2l\delta+6)}{6(27-10\delta)(l-1)}. \tag{37}$$

The right-hand side of (37) is non-negative if

$$l \leq \bar{l}_2(\delta) = \frac{6(1+\delta)}{3+2\delta}. \tag{38}$$

As we consider $l \geq \bar{l}_2(\delta)$, the optimal deviation price of firm B follows from $\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) = (l-2)/[2(l-1)]$. Then from the continuity of firm B's profits we conclude that it does not have an incentive to deviate.

Deviation on $\alpha(\cdot) \geq 1/(l-1)$.

i) The incentives of firm A. If firm A deviates, then its profit is given by (26). Keeping p_B^x at p_B^{x*} in (24) and taking the derivative of (26) with respect to p_A^x yields the deviation price of firm A:

$$p_A^{x,dev}\left(p_B^{x*}\right) = -\frac{567\delta - 972l - 621l\delta - 234\delta^2 + 240\delta^3 + 318l\delta^2 - 80l\delta^3 + 486}{-300\delta^2 + 270\delta + 1458}. \tag{39}$$

Note that for any δ and $l \geq 2$ it holds that $p_A^{x,dev}\left(p_B^{x*}\right) > 0$. The prices $p_A^{x,dev}\left(p_B^{x*}\right)$ and p_B^{x*} yield the market share of firm B in the first period:

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) = \frac{297\delta - 162l - 189l\delta - 212\delta^2 + 114l\delta^2 + 324}{\left(100\delta^2 - 90\delta - 486\right)(l-1)}.$$

Note that for any δ and any $l \geq 2$ it holds that $\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) < 1$. The other comparison shows that

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) - \frac{1}{l-1} = -\frac{3\left(129\delta - 54l - 63l\delta - 104\delta^2 + 38l\delta^2 + 270\right)}{2(l-1)\left(-50\delta^2 + 45\delta + 243\right)}. \tag{40}$$

The right-hand side of (40) is non-negative if

$$l \geq \bar{l}_5(\delta) := \frac{129\delta - 104\delta^2 + 270}{63\delta - 38\delta^2 + 54}. \tag{41}$$

Solving $\bar{l}_5(\delta) = \bar{l}_3(\delta)$, we arrive at $\delta \approx 0.89$. If $\delta < 0.89$, then $\bar{l}_5(\delta) > \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$, such that (41) does not hold and $\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) < 1/(l-1)$.

Consider first $\delta < 0.89$ and $\delta \geq 0.89$ with $l \leq \bar{l}_5(\delta)$. The optimal deviation price of firm A follows from $\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) = 1/(l-1)$, which yields

$$p_A^{x,dev}\left(p_B^{x*}\right) = \frac{27\delta - 27l + 12l\delta + 24\delta^2 - 8l\delta^2 - 108}{30\delta - 81}.$$

Note that for any l and any δ, it holds that $p_A^{x,dev}\left(p_B^{x*}\right) > 0$. Then the difference between the equilibrium and deviation profits of firm A is

$$\frac{l^2\left(112\delta^3 - 480\delta^2 + 513\delta + 243\right) - l\left(712\delta^3 - 2874\delta^2 + 2322\delta + 2430\right) + 1053\delta^3 - 3807\delta^2 + 1215\delta + 6075}{3(10\delta - 27)^2}. \tag{42}$$

The numerator of (42) is a quadratic function with respect to l with a non-positive discriminant (for any δ), such that it does not have roots. As this function opens upwards (for any δ), it takes only positive values. Hence, firm A does not have an incentive to deviate if $\delta < 0.89$ or if

$\delta \geq 0.89$ with $l \leq \bar{l}_5(\delta)$ hold.

Consider now $\delta \geq 0.89$ with $\bar{l}_5(\delta) < l < \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$, in which case (41) holds and (39) is the optimal deviation price of firm A. The difference between the equilibrium and deviation profits of firm A is

$$\frac{\delta\left[-l^2\left(2092\delta^3-8796\delta^2+6615\delta-2916\right)+l\left(9472\delta^3-36876\delta^2+10530\delta+11664\right)-11388\delta^3+42264\delta^2+5805\delta-43740\right]}{12(5\delta+9)(10\delta-27)^2}. \quad (43)$$

The function in the parentheses of (43) is quadratic in l, opens upwards for any δ and has two roots, one of which is negative for any δ, while the other root is smaller than $\bar{l}_5(\delta)$ for any δ. Hence, for any $\bar{l}_5(\delta) < l < \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$ and $\delta \geq 0.89$, the function in the parentheses in (43) is positive, such that firm A does not have an incentive to deviate.

ii) The incentives of firm B. Given the rival's price, p_A^{x*} in (24), there exists $p_B^{x,dev} \geq 0$, which yields the share of firm B in the first period satisfying $\alpha(\cdot) \geq 1/(l-1)$ only if

$$l \geq l_5(\delta) := \frac{-120\delta+24\delta^2+189}{-36\delta+8\delta^2+54}. \quad (44)$$

In the following we will restrict attention to l satisfying (44), because only in that case firm B can deviate. Keeping p_A^x at p_A^{x*} in (24) and taking the derivative of (27) with respect to p_B^x yields the deviation price of firm B:

$$p_B^{x,dev}\left(p_A^{x*}\right) = \frac{243l+594\delta-216l\delta-228\delta^2+24\delta^3+72l\delta^2-8l\delta^3-486}{30\delta^2-351\delta+729}.$$

Note that $p_B^{x,dev}\left(p_A^{x*}\right) > 0$ for any δ and $l > 2$. The prices $p_B^{x,dev}\left(p_A^{x*}\right)$ and p_A^{x*} yield the market share of firm B in the first period:

$$\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) = \frac{p_A^{x*}-p_B^{x,dev}\left(p_A^{x*}\right)-1}{l-1} = \frac{81l+108\delta-54l\delta-26\delta^2+12l\delta^2-162}{(l-1)\left(10\delta^2-117\delta+243\right)}. \quad (45)$$

Note that for any l and δ it holds that $\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) < 1$. The other comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) - \frac{1}{l-1} = \frac{3\left(27l+75\delta-18l\delta-12\delta^2+4l\delta^2-135\right)}{(l-1)\left(10\delta^2-117\delta+243\right)}. \quad (46)$$

The right-hand side of (46) is non-negative if

$$l \geq \frac{135-75\delta+12\delta^2}{27-18\delta+4\delta^2}. \quad (47)$$

Note that the right-hand side of (47) is for any δ larger than 4, such that (47) does not hold and $\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) < 1/(l-1)$ stated in (45). Hence, the optimal deviation price of firm B is such that $\alpha(\cdot) = 1/(l-1)$, which yields

$$p_B^{x,dev}\left(p_A^{x*}\right) = \frac{54l+120\delta-36l\delta-24\delta^2+8l\delta^2-189}{81-30\delta}. \tag{48}$$

Note that for any $l > l_5(\delta)$, $p_B^{x,dev}\left(p_A^{x*}\right) > 0$. Then the optimal deviation price of firm B is given by (48) and the difference between the equilibrium and deviation profits is

$$-\frac{l^2\left(52\delta^3-204\delta^2+297\delta-243\right)-l\left(352\delta^3-1608\delta^2+2862\delta-2430\right)+588\delta^3-3123\delta^2+6642\delta-6075}{3(10\delta-27)^2}. \tag{49}$$

Note that the numerator of (49) is a quadratic function with respect to l, which opens downwards (for any δ). As its discriminant is non-positive (for any δ), for any δ and l it takes only negative values, such that the whole term in (49) is positive for any δ and l and firm B does not have an incentive to deviate.

Conclusion from 1.a). We conclude that on $\bar{l}_2(\delta) \leq l \leq \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$ the equilibrium (24) exists if $l \geq l_2(\delta)$, with

$$l_2(\delta) = \frac{-120\delta^3+276\delta^2+6552\delta+6156+36(27-10\delta)(1+\delta)\sqrt{(\delta+1)(9-\delta)}}{2\left(-100\delta^3+372\delta^2+531\delta+2268\right)}.$$

1.b) Consider now the equilibrium (15).

Deviation on $\alpha \leq (l-2)/[2(l-1)]$.

i) The incentives of firm A. If firm A deviates, then its profits are given by (20). Maximizing (20) with respect to p_A^x and keeping p_B^x at p_B^{x*} in (15) yields the deviation price of firm A:

$$p_A^{x,dev}\left(p_B^{x*}\right) = \frac{36l+22\delta+13l\delta+20\delta^2-10l\delta^2-18}{-4\delta^2+30\delta+54}, \tag{50}$$

such that the market share of firm B in the first period is

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) = \frac{p_A^{x,dev}\left(p_B^{x*}\right)-p_B^{x*}-1}{l-1} = \frac{18l+6\delta-3l\delta+22\delta^2-8l\delta^2-36}{(l-1)\left(54-4\delta^2+30\delta\right)}.$$

Note that for any l and any δ, $p_A^{x,dev}\left(p_B^{x*}\right)$ in (50) is positive. The comparison shows that

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) - \frac{l-2}{2(l-1)} = \frac{3\left(12\delta-3l-6l\delta+6\delta^2-2l\delta^2+6\right)}{2(l-1)\left(-2\delta^2+15\delta+27\right)}. \tag{51}$$

Note that for any $l \geq \bar{l}_2(\delta)$ it holds that

$$l > \frac{12\delta+6\delta^2+6}{6\delta+2\delta^2+3}, \tag{52}$$

such that the right-hand side of (51) is negative. Note next that $\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) \geq 0$ if

$$l \geq \frac{36-6\delta-22\delta^2}{18-3\delta-8\delta^2}. \tag{53}$$

As any $l \geq 2$ also satisfies (53), $\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) \geq 0$ holds. We conclude that (50) is the optimal deviation price of firm A.

Given (50), the difference between the equilibrium and the deviation profits of firm A is

$$\begin{aligned} & -\frac{\delta f_2(l,\delta)}{4(9-\delta)(2\delta+3)^2}, \text{ with} \\ f_2(l,\delta) \quad : & = l^2\left(4\delta^3+24\delta^2+49\delta+28\right) - l\left(24\delta^3+124\delta^2+176\delta+76\right) \\ & +40\delta^3+120\delta^2+120\delta+40. \end{aligned}$$

The function $f_2(l,\delta)$ opens upwards (for any δ) and has two roots:

$$\begin{aligned} l_3(\delta) \quad : & = \frac{24\delta^3+124\delta^2+176\delta+76-4(\delta+1)(2\delta+3)\sqrt{(\delta+1)(9-\delta)}}{2\left(4\delta^3+24\delta^2+49\delta+28\right)}, \\ l_4(\delta) \quad : & = \frac{24\delta^3+124\delta^2+176\delta+76+4(\delta+1)(2\delta+3)\sqrt{(\delta+1)(9-\delta)}}{2\left(4\delta^3+24\delta^2+49\delta+28\right)}. \end{aligned} \tag{54}$$

For any δ it holds that $l_3(\delta) < 2$ and $\bar{l}_2(\delta) \leq l_4(\delta) \leq \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$. It then follows that for any l such that $\bar{l}_2(\delta) \leq l \leq \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$, firm A does not have an incentive to deviate if $l \leq l_4(\delta)$ and deviates otherwise.

ii) The incentives of firm B. If firm B deviates, then its profits are given by (21). Maximizing (21) with respect to p_B^x and keeping p_A^x at p_A^{x*} in (15), yields the deviation price of firm B:

$$p_B^{x,dev}\left(p_A^{x*}\right) = -\frac{5\delta-9l-7l\delta+4l\delta^2+18}{-8\delta^2+6\delta+27}.$$

Note that $p_B^{x,dev}\left(p_A^{x*}\right) \geq 0$ for any δ and any $l \geq 2$. This deviation price yields the market share of firm B in the first period:

$$\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) = \frac{p_A^{x*} - p_B^{x,dev}\left(p_A^{x*}\right) - 1}{l-1} = \frac{9l + 3\delta + 3l\delta + 8\delta^2 - 4l\delta^2 - 18}{(l-1)\left(-8\delta^2 + 6\delta + 27\right)}.$$

The comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) - \frac{l-2}{2(l-1)} = \frac{9(2\delta - l + 2)}{2(l-1)\left(-8\delta^2 + 6\delta + 27\right)}. \tag{55}$$

The right-hand side of (55) is non-positive if $l \geq \bar{l}_3(\delta) = 2(1+\delta)$. Hence, for any $\bar{l}_2(\delta) \leq l \leq \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$, the optimal deviation price of firm B follows from $\alpha(\cdot) = (l-2)/[2(l-1)]$. Then from the continuity of firm B's profits we conclude that for any l and δ firm B does not have an incentive to deviate.

Deviation on $\alpha \geq 1/(l-1)$.

i) The incentives of firm A. If firm A deviates, it realizes the profit (26). Keeping p_B^x at p_B^{x*} in (15) and taking the derivative of (26) with respect to p_A^x yields the deviation price of firm A:

$$p_A^{x,dev}\left(p_B^{x*}\right) = -\frac{41\delta - 36l - 64l\delta + 18\delta^2 - 28l\delta^2 + 18}{20\delta^2 + 66\delta + 54}, \tag{56}$$

which is non-negative if

$$l \geq \frac{41\delta + 18\delta^2 + 18}{64\delta + 28\delta^2 + 36}. \tag{57}$$

Note that for any δ the right-hand side of (57) is smaller than 1, such that (57) is fulfilled as strict inequality for any δ and l. The price (56) yields the market share of firm B in the first period:

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) = \frac{p_A^{x,dev}\left(p_B^{x*}\right) - p_B^{x*} - 1}{l-1} = -\frac{69\delta - 18l - 36l\delta + 28\delta^2 - 18l\delta^2 + 36}{(l-1)\left(20\delta^2 + 66\delta + 54\right)}. \tag{58}$$

The comparison shows that

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) - \frac{1}{l-1} = -\frac{3\left(45\delta - 6l - 12l\delta + 16\delta^2 - 6l\delta^2 + 30\right)}{2(l-1)\left(10\delta^2 + 33\delta + 27\right)}. \tag{59}$$

The right-hand side of (59) is non-negative if

$$l \geq \frac{45\delta + 16\delta^2 + 30}{12\delta + 6\delta^2 + 6}. \tag{60}$$

We showed above that on $\bar{l}_2(\delta) \leq l \leq \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$, the equilibrium (15) exists if $l \leq l_4(\delta)$, the latter being stated in (54). The comparison shows that for any δ, the right-hand side of (60) is larger than $l_4(\delta)$, such that (60) is not fulfilled for any δ and the right-hand side of (59) is negative. Hence, the optimal deviation price of firm A follows from $\alpha\left(p_A^{x,dev}(p_B^{x*}), p_B^{x*}\right) = 1/(l-1)$. Then from the continuity of firm A's profits, we conclude that firm A does not have an incentive to deviate.

ii) The incentives of firm B. Keeping p_A^x at p_A^{x*} in (15) and requiring that $\alpha \geq 1/(l-1)$ yields the restriction on firm B's deviation price:

$$p_B^{x,dev} \leq -\tfrac{4\delta-2l-2l\delta+7}{2\delta+3}. \tag{61}$$

There exists $p_B^{x,dev} \geq 0$, which satisfies (61) if $l \geq (4\delta+7)/(2\delta+2)$. We showed above that on $\bar{l}_2(\delta) \leq l \leq \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$, the equilibrium (15) exists if $l \leq l_4(\delta)$. Note that for any δ it holds that $(4\delta+7)/(2\delta+2) > l_4(\delta)$, which implies that firm B cannot deviate.

Conclusion from 1.b). We conclude that on $\bar{l}_2(\delta) \leq l \leq \min\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$, the equilibrium (15) exists if $l \leq l_4(\delta)$, with

$$l_4(\delta) = \tfrac{24\delta^3+124\delta^2+176\delta+76+4(\delta+1)(2\delta+3)\sqrt{(\delta+1)(9-\delta)}}{2\left(4\delta^3+24\delta^2+49\delta+28\right)}.$$

2) Consider now $2 \leq l \leq \bar{l}_2(\delta)$, which yields the unique candidate equilibrium (15).

Deviation on $\alpha \leq (l-2)/[2(l-1)]$.

i) The incentives of firm A. Refer to the analysis in *1.b)*. For any δ it holds that

$$2 \leq \tfrac{12\delta+6\delta^2+6}{6\delta+2\delta^2+3} \leq \bar{l}_2(\delta).$$

For any

$$\tfrac{12\delta+6\delta^2+6}{6\delta+2\delta^2+3} \leq l \leq \bar{l}_2(\delta)$$

the analysis in *1.b)* applies, such that firm A does not have an incentive to deviate. Consider now

$$2 \leq l \leq \tfrac{12\delta+6\delta^2+6}{6\delta+2\delta^2+3},$$

in which case the optimal deviation price of firm A follows from

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) = \tfrac{l-2}{2(l-1)}.$$

Then from the continuity of firm A's profits, we conclude that firm A does not have an incentive to deviate either.

ii) The incentives of firm B. The analysis in *1.b)* allows us to conclude that firm B does not have an incentive to deviate.

Deviation on $\alpha \geq 1/(l-1)$. The analysis in *1.b)* allows us to conclude that neither firm A, nor firm B have an incentive to deviate.

Conclusion from 2). We conclude that on $2 \leq l \leq \bar{l}_2(\delta)$ the unique equilibrium is (15).

3) Consider now $\bar{l}_3(\delta) \leq l \leq \min\left\{4, \bar{l}_4(\delta)\right\}$, which yields the unique candidate equilibrium (24).

Deviation on $(l-2)/[2(l-1)] \leq \alpha(\cdot) \leq 1/(l-1)$.

i) The incentives of firm A. We introduce a new notation:

$$l_6(\delta) := \tfrac{48\delta - 6\delta^2 + 54}{8\delta^2 - 12\delta + 27}.$$

Consider first $\bar{l}_3(\delta) \leq l \leq \min\left\{l_6(\delta), \bar{l}_4(\delta)\right\}$. From the analysis in *1.a)* it follows that firm A does not have an incentive to deviate. Consider now $l_6(\delta) \leq l \leq \min\left\{\bar{l}_4(\delta), 4\right\}$. From the analysis in *1.a)* we conclude that the optimal deviation price of firm A follows from $(l-2)/[2(l-1)] = \alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right)$. As firm A's profits are continuous, we conclude that firm A does not have an incentive to deviate.

ii) The incentives of firm B. From the analysis in *1.a)* we conclude that firm B does not have an incentive to deviate.

Deviation on $\alpha(\cdot) \geq 1/(l-1)$.

i) The incentives of firm A. We first consider $\bar{l}_3(\delta) \leq l \leq \min\left\{\bar{l}_5(\delta), 4\right\}$, with $\bar{l}_5(\delta)$ being defined in (41). From the analysis in *1.a)* it follows that firm A does not have an incentive to deviate. Consider now $\bar{l}_5(\delta) \leq l \leq \min\left\{\bar{l}_4(\delta), 4\right\}$. From the analysis in *1.a)* it follows that firm A does not have an incentive to deviate either.

ii) The incentives of firm B. From the analysis in *1.a)* it follows that firm B does not have an incentive to deviate.

Conclusion from 3). We conclude that on $\bar{l}_3(\delta) \leq l \leq \min\left\{4, \bar{l}_4(\delta)\right\}$ the unique equilibrium is (24).

4) Consider now $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\} \leq l \leq 4$, which yields two candidate equilibria, (24) and (28).

4.a) Consider first the candidate equilibrium (24).

Deviation on $\alpha(\cdot) \geq 1/(l-1)$.

i) The incentives of firm A. If firm A deviates, its profit is given by (26). Keeping p_B^x at p_B^{x*} in (24) and taking the derivative of (26) with respect to p_A^x yields the deviation price of firm A:

$$p_A^{x,dev}(p_B^{x*}) = -\frac{-l\left(80\delta^3-318\delta^2+621\delta+972\right)+240\delta^3-234\delta^2+567\delta+486}{270\delta-300\delta^2+1458}, \tag{62}$$

which is non-negative if

$$l \geq \frac{567\delta-234\delta^2+240\delta^3+486}{621\delta-318\delta^2+80\delta^3+972}. \tag{63}$$

Since the right-hand side of (63) is smaller than 1 for any δ, then (63) holds for any l. Using p_B^{x*} in (24) and $p_A^{x,dev}(p_B^{x*})$ in (62) we calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{x,dev}(p_B^{x*}), p_B^{x*}\right) = -\frac{297\delta-162l-189l\delta-212\delta^2+114l\delta^2+324}{(l-1)\left(486-100\delta^2+90\delta\right)}.$$

The comparison shows that $\alpha\left(p_A^{x,dev}(p_B^{x*}), p_B^{x*}\right) \geq 1/(l-1)$ if

$$l \geq \frac{129\delta-104\delta^2+270}{63\delta-38\delta^2+54},$$

which is fulfilled for any δ and any $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\} \leq l \leq 4$. The other comparison shows that $\alpha\left(p_A^{x,dev}(p_B^{x*}), p_B^{x*}\right) \leq 1$ if

$$l \geq \frac{-207\delta+112\delta^2+162}{-99\delta+14\delta^2+324}. \tag{64}$$

Since the right-hand side of (64) is smaller than 1 for any δ, then (64) holds for any l and any δ. Hence, (62) is the optimal deviation price of firm A, which yields the following difference between the equilibrium and the deviation profits:

$$\frac{\delta\left[-l^2\left(2092\delta^3-8796\delta^2+6615\delta-2916\right)+l\left(9472\delta^3-36876\delta^2+10530\delta+11664\right)-11388\delta^3+42264\delta^2+5805\delta-43740\right]}{12(5\delta+9)(10\delta-27)^2}. \quad (65)$$

The function in the brackets in (65) is quadratic in l. It opens upwards (for any δ) and has two roots:

$$\frac{-\left(9472\delta^3-36\,876\delta^2+10\,530\delta+11\,664\right)+4(27-10\delta)\sqrt{-\left(3485\delta^4-16\,032\delta^3+801\delta^2+42\,930\delta-55\,404\right)}}{-2\left(2092\delta^3-8796\delta^2+6615\delta-2916\right)},$$

$$\frac{-\left(9472\delta^3-36\,876\delta^2+10\,530\delta+11\,664\right)-4(27-10\delta)\sqrt{-\left(3485\delta^4-16\,032\delta^3+801\delta^2+42\,930\delta-55\,404\right)}}{-2\left(2092\delta^3-8796\delta^2+6615\delta-2916\right)}.$$

Both of these roots are smaller than $\max\left\{\bar{l}_3(\delta),\bar{l}_4(\delta)\right\}$ for any δ. Hence, for any δ and any $\max\left\{\bar{l}_3(\delta),\bar{l}_4(\delta)\right\} \leq l \leq 4$, the expression in (65) is non-negative, such that firm A does not have an incentive to deviate.

ii) The incentives of firm B. If firm B deviates, then its profit is given by (27). Keeping p_A^x at p_A^{x*} in (24) and taking the derivative of (27) with respect to p_B^x yields the deviation price of firm B:

$$p_B^{x,dev}\left(p_A^{x*}\right) = \frac{243l+594\delta-216l\delta-228\delta^2+24\delta^3+72l\delta^2-8l\delta^3-486}{30\delta^2-351\delta+729}, \quad (66)$$

which is non-negative for any

$$l \geq \frac{594\delta-228\delta^2+24\delta^3-486}{216\delta-72\delta^2+8\delta^3-243}. \quad (67)$$

Since the right-hand side of (67) is for any δ not lager than 2, then (67) holds for any δ and any $l \geq 2$. Using $p_B^{x,dev}\left(p_A^{x*}\right)$ in (66), we can calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) = \frac{81l+108\delta-54l\delta-26\delta^2+12l\delta^2-162}{(l-1)\left(10\delta^2-117\delta+243\right)}.$$

The comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) - \frac{1}{l-1} = \frac{3\left(27l+75\delta-18l\delta-12\delta^2+4l\delta^2-135\right)}{(l-1)\left(10\delta^2-117\delta+243\right)}. \quad (68)$$

The right-hand side of (68) is non-negative if

$$l \geq \frac{-75\delta+12\delta^2+135}{-18\delta+4\delta^2+27}. \quad (69)$$

Since the right-hand side of (69) is not smaller than 5 for any δ, (69) does not hold for any δ and any $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\} \leq l \leq 4$. Hence, the optimal deviation price of firm B follows from

$$\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) = \frac{1}{l-1}$$

and is given by

$$p_B^{x,dev}\left(p_A^{x*}\right) = \frac{54l+120\delta-36l\delta-24\delta^2+8l\delta^2-189}{81-30\delta}, \tag{70}$$

which is non-negative if

$$l \geq \frac{-120\delta+24\delta^2+189}{-36\delta+8\delta^2+54}. \tag{71}$$

The right-hand side of (71) is for any δ smaller than $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$, such that (71) holds for any δ and any $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\} \leq l \leq 4$. Hence, $p_B^{x,dev}\left(p_A^{x*}\right)$ in (70) is the optimal deviation price of firm B. Using the latter we calculate the difference between the equilibrium and the deviation profits of firm B:

$$-\frac{l^2\left(52\delta^3-204\delta^2+297\delta-243\right)-l\left(352\delta^3-1608\delta^2+2862\delta-2430\right)+588\delta^3-3123\delta^2+6642\delta-6075}{3(10\delta-27)^2}. \tag{72}$$

The numerator of (72) is a quadratic function of l, which opens downwards and has a non-positive discriminant (for any δ). It follows that for any δ and any l the expression in (72) is non-negative, such that firm B does not have an incentive to deviate.

Deviation on $(l-2)/[2(l-1)] \leq \alpha(\cdot) \leq 1/(l-1)$.

i) The incentives of firm A. If firm A deviates, then its profit is given by (13). Keeping p_B^x at p_B^{x*} in (24) and taking the derivative of (13) with respect to p_A^x yields the deviation price of firm A:

$$p_A^{x,dev}\left(p_B^{x*}\right) = \frac{108l+6\delta+93l\delta+12\delta^2-48\delta^3-46l\delta^2+16l\delta^3-54}{-60\delta^2+102\delta+162}, \tag{73}$$

which is non-negative if

$$l \geq -\frac{6\delta+12\delta^2-48\delta^3-54}{93\delta-46\delta^2+16\delta^3+108}. \tag{74}$$

As the right-hand side of (74) is for any δ smaller than 1, (74) holds for any δ and any l. Using (73), we calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) = -\frac{54\delta-54l-63l\delta-54\delta^2+38l\delta^2+108}{(l-1)\left(162-60\delta^2+102\delta\right)}.$$

The comparison shows that

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right),p_B^{x*}\right)-\frac{1}{l-1}=-\frac{\left(156\delta-54l-63l\delta-114\delta^2+38l\delta^2+270\right)}{6(l-1)\left(-10\delta^2+17\delta+27\right)}. \tag{75}$$

The right-hand side of (75) is non-positive if

$$l\leq l_7\left(\delta\right):=\frac{156\delta-114\delta^2+270}{63\delta-38\delta^2+54}. \tag{76}$$

The other comparison shows that

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right),p_B^{x*}\right)-\frac{l-2}{2(l-1)}=\frac{48\delta-27l+12l\delta-6\delta^2-8l\delta^2+54}{6(l-1)\left(-10\delta^2+17\delta+27\right)},$$

which is non-negative if

$$l\leq l_8\left(\delta\right):=\frac{48\delta-6\delta^2+54}{-12\delta+8\delta^2+27}. \tag{77}$$

Note that if $\delta=\left(24-3\sqrt{7}\right)/19\approx 0.85$, then $l_7\left(\delta\right)=l_8\left(\delta\right)=4$. Consider $\max\left\{l_8\left(\delta\right),\bar{l}_4\left(\delta\right)\right\}\leq l\leq 4$, in which case (77) does not hold and $p_A^{x,dev}\left(p_B^{x*}\right)$ follows from $\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right),p_B^{x*}\right)=(l-2)/\left[2(l-1)\right]$. As firm A's profit is continuous, we conclude that firm A does not have an incentive to deviate. Consider now $\max\left\{l_7\left(\delta\right),\bar{l}_3\left(\delta\right)\right\}\leq l\leq 4$, in which case (76) does not hold and $p_A^{x,dev}\left(p_B^{x*}\right)$ follows from $\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right),p_B^{x*}\right)=1/(l-1)$. As firm A's profit is continuous, we conclude that firm A does not have an incentive to deviate. Consider finally $\max\left\{\bar{l}_3\left(\delta\right),\bar{l}_4\left(\delta\right)\right\}\leq l\leq\min\left\{l_7\left(\delta\right),l_8\left(\delta\right)\right\}$, in which case both (76) and (77) are fulfilled and the optimal deviation price of firm A is given by (73). Using the latter price we calculate the difference between the equilibrium and the deviation profits of firm A:

$$\frac{\delta\left[l^2\left(-100\delta^3+372\delta^2+531\delta+2268\right)-l\left(-120\delta^3+276\delta^2+6552\delta+6156\right)-360\delta^3+2520\delta^2+6120\delta+3240\right]}{36(\delta+1)(10\delta-27)^2}. \tag{78}$$

For any δ, the function in the brackets in (78) is quadratic in l, looks upwards and has two roots both of which are smaller than $\max\left\{\bar{l}_3\left(\delta\right),\bar{l}_4\left(\delta\right)\right\}$. Hence, the expression in (78) is non-negative for any δ and any $\max\left\{\bar{l}_3\left(\delta\right),\bar{l}_4\left(\delta\right)\right\}\leq l\leq\min\left\{l_7\left(\delta\right),l_8\left(\delta\right)\right\}$, such that firm A does not have an incentive to deviate.

i) The incentives of firm B. If firm B deviates, then its profits are given by (14). Keeping p_A^x at p_A^{x*} in (24) and taking the derivative of (14) with respect to p_B^x yields the deviation price of firm B:

$$p_B^{x,dev}\left(p_A^{x*}\right) = \frac{27l+45\delta-18l\delta-12\delta^2+4l\delta^2-54}{81-30\delta}, \tag{79}$$

which is non-negative if

$$l \geq \frac{-45\delta+12\delta^2+54}{-18\delta+4\delta^2+27}. \tag{80}$$

Since the right-hand side of (80) is for any δ not larger than 2, then (80) holds for any δ and any $l \geq 2$. Using (79) we can calculate the equilibrium share of firm B in the first period:

$$\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) = \frac{27l+45\delta-18l\delta-12\delta^2+4l\delta^2-54}{(l-1)(81-30\delta)}.$$

The comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) - \frac{l-2}{2(l-1)} = \frac{(9-4\delta)(6\delta-3l-2l\delta+6)}{6(27-10\delta)(l-1)}. \tag{81}$$

The right-hand side of (81) is non-negative if

$$l \leq \bar{l}_2(\delta) = \frac{6(1+\delta)}{2\delta+3}. \tag{82}$$

Note that for any δ and any $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\} \leq l \leq 4$, (82) does not hold, such that the optimal deviation price of firm B follows from $\alpha\left(p_A^{x*}, p_B^{x,dev}\left(p_A^{x*}\right)\right) = (l-2)/[2(l-1)]$. As firm B's profits are continuous, we conclude that it does not have an incentive to deviate.

Conclusion from 4.a) We conclude that for any δ and any $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\} \leq l \leq 4$ there exists the equilibrium (24).

4.b) Consider now the candidate equilibrium (28).

Deviation on $\alpha(\cdot) \leq (l-2)/[2(l-1)]$.

i) The incentives of firm A. If firm A deviates, its profit is given by (20). Keeping p_B^x at p_B^{x*} in (28) and taking the derivative of (20) with respect to p_A^x yields the deviation price of firm A:

$$p_A^{x,dev}\left(p_B^{x*}\right) = \frac{l\left(12\delta^3-192\delta^2+27\delta+972\right)-32\delta^3+396\delta^2+675\delta-486}{-48\delta^2+270\delta+1458}. \tag{83}$$

The price $p_A^{x,dev}\left(p_B^{x*}\right)$ is non-negative if

$$l \geq -\frac{675\delta+396\delta^2-32\delta^3-486}{27\delta-192\delta^2+12\delta^3+972}. \tag{84}$$

The right-hand side of (84) is for any δ smaller than 1, such that (84) holds for any δ and any l and $p_A^{x,dev}\left(p_B^{x*}\right) \geq 0$. Using $p_A^{x,dev}\left(p_B^{x*}\right)$ in (83) we can calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) = -\frac{-l\left(14\delta^2+99\delta-162\right)+40\delta^2+207\delta-324}{-(l-1)\left(-16\delta^2+90\delta+486\right)}. \tag{85}$$

The numerator of (85) is non-negative if

$$l \geq \frac{207\delta+40\delta^2-324}{99\delta+14\delta^2-162}. \tag{86}$$

The right-hand side of (86) is for any δ smaller or equal to 2, such that (86) is fulfilled for any δ and any $l \geq 2$. The denominator of (86) is negative for any δ and any l. It then follows that for any δ and any $l \geq 2$, $\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) \geq 0$ holds.

The comparison shows that

$$\alpha\left(p_A^{x,dev}\left(p_B^{x*}\right), p_B^{x*}\right) - \frac{l-2}{2(l-1)} = \frac{3\left[-l\left(2\delta^2+48\delta+27\right)+8\delta^2+99\delta+54\right]}{2(l-1)\left(-8\delta^2+45\delta+243\right)}. \tag{87}$$

The right-hand side of (87) is non-positive if

$$l \geq \frac{99\delta+8\delta^2+54}{48\delta+2\delta^2+27}. \tag{88}$$

The right-hand side of (88) is for any δ smaller than $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$. Hence, for any δ and any $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\} \leq l \leq 4$, (88) holds. We conclude that $p_A^{x,dev}\left(p_B^{x*}\right)$ in (83) is the optimal deviation price of firm A. Using the latter price we can calculate the difference between the equilibrium and the deviation profits of firm A:

$$\frac{\delta\left[-264l^2\delta^3+1116l^2\delta^2-2619l^2\delta-2916l^2+1184l\delta^3-2676l\delta^2+2754l\delta-11\,664l-1472\delta^3+1200\delta^2+9585\delta+43\,740\right]}{12(9-\delta)(8\delta+27)^2}. \tag{89}$$

The expression in the brackets in the numerator of (89) is a quadratic function with respect to l, which opens downwards (for any δ) and has two roots, one of which is negative for any δ and the other one is smaller than $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\}$ for any δ:

$$\frac{-\left(1184\delta^3-2676\delta^2+2754\delta-11\,664\right)+4(8\delta+27)\sqrt{-149\delta^4+2472\delta^3-12\,033\delta^2+10\,530\delta+55\,404}}{-2\left(264\delta^3-1116\delta^2+2619\delta+2916\right)}, \tag{90}$$
$$\frac{-\left(1184\delta^3-2676\delta^2+2754\delta-11\,664\right)-4(8\delta+27)\sqrt{-149\delta^4+2472\delta^3-12\,033\delta^2+10\,530\delta+55\,404}}{-2\left(264\delta^3-1116\delta^2+2619\delta+2916\right)}.$$

It then follows that for any δ and any $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\} \leq l \leq 4$ the expression in the brackets in (89) is negative, which implies that firm A has an incentive to deviate.

Conclusion from 4.b) We conclude that for any δ and any $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\} \leq l \leq 4$ there does not exist the equilibrium (28).

Conclusion from 4). We conclude that for any δ and any $\max\left\{\bar{l}_3(\delta), \bar{l}_4(\delta)\right\} \leq l \leq 4$ there exists the unique equilibrium (24).

5) Consider finally $\bar{l}_4(\delta) \leq l \leq \bar{l}_3(\delta)$, which yields three candidate equilibria, (15), (24) and (28).

5.a) Consider first the candidate equilibrium (28).

Deviation on $\alpha(\cdot) \leq (l-2)/[2(l-1)]$.

i) The incentives of firm A. Note that the right-hand side of (88) is for any δ smaller than $\bar{l}_4(\delta)$. Note also that the second expression in (90) is for any δ smaller than $\bar{l}_4(\delta)$. Then similar to the analysis in *4.b)* we conclude that for any δ and any $\bar{l}_4(\delta) \leq l \leq \bar{l}_3(\delta)$, firm A has an incentive to deviate.

Conclusion from 5.a). We conclude that for any δ and any $\bar{l}_4(\delta) \leq l \leq \bar{l}_3(\delta)$, the equilibrium (28) does not exist.

5.b) We consider now the candidate equilibrium (15).

Deviation on $\alpha \leq (l-2)/[2(l-1)]$.

i) The incentives of firm A. From the analysis in *1.b)* it follows that for any δ and any $\bar{l}_4(\delta) \leq l \leq \bar{l}_3(\delta)$, firm A has an incentive to deviate.

Conclusion from 5.b). We conclude that for any δ and any $\bar{l}_4(\delta) \leq l \leq \bar{l}_3(\delta)$, the equilibrium (15) does not exist.

5.c) We consider finally the candidate equilibrium (24).

i) The incentives of firm B. We can use the results of the analysis in *1.a)* to conclude that firm B does not have an incentive to deviate on $(l-2)/[2(l-1)] \leq \alpha(\cdot) \leq 1/(l-1)$. Note also that for any δ it holds that $l_5(\delta) < \bar{l}_4(\delta)$, with $l_5(\delta)$ being defined in (44). Then we can use the results of the analysis in *1.a)* to conclude that firm B does not have an incentive to deviate on $\alpha(\cdot) \geq 1/(l-1)$ either.

ii) The incentives of firm A.

Deviation on $\alpha(\cdot) \geq 1/(l-1)$**.** Note that for any δ it holds that $\bar{l}_5(\delta) \leq \bar{l}_4(\delta)$, with $\bar{l}_5(\delta)$ being defined in (41). Based on this result we conclude from the analysis in *1.a)* that for any δ and any $\bar{l}_4(\delta) \leq l \leq \bar{l}_3(\delta)$, firm A does not have an incentive to deviate.

Deviation on $(l-2)/[2(l-1)] \leq \alpha(\cdot) \leq 1/(l-1)$**.** We first note that the right-hand side of (33) is for any δ not smaller than $\bar{l}_3(\delta)$. Consider first δ and l, for which (35) holds. We can then use the results of the analysis in *1.a)* to conclude that for any δ and any $\bar{l}_4(\delta) \leq l \leq \bar{l}_3(\delta)$, firm A does not have an incentive to deviate. Consider next δ and l, for which (35) does not hold, such that $p_A^{x,dev}(p_B^{x*})$ follows from $\alpha\left(p_A^{x,dev}(p_B^{x*}), p_B^{x*}\right) = 1/(l-1)$. We showed, however, above that firm A does not have an incentive to deviate on $\alpha(\cdot) \geq 1/(l-1)$ for any δ and any $\bar{l}_4(\delta) \leq l \leq \bar{l}_3(\delta)$.

Conclusion from 5.c). We conclude that for any δ and any $\bar{l}_4(\delta) \leq l \leq \bar{l}_3(\delta)$, the equilibrium (24) exists.

Conclusion from 5. We conclude that for any δ and any $\bar{l}_4(\delta) \leq l \leq \bar{l}_3(\delta)$, the unique equilibrium is (24).

Part 3. Consider finally $l \geq 4$, in which case second-period profits are given by different functions depending on α.

i) Consider first $\alpha \leq 1/(l-1)$. The profits of the second period are given by (8). Firm A chooses p_A^x to maximize the profit (20). Firm B chooses p_B^x to maximize the profit (21). Solving simultaneously firms' first-order conditions yields the equilibrium prices (24), both of which are positive for any δ and any $l \geq 4$. These prices yield the market share of firm B in the first period given by (25), which is positive for any δ and any $l \geq 4$. The comparison shows that

$$\alpha(p_A^{x*}, p_B^{x*}) - \tfrac{1}{l-1} = \tfrac{9l+29\delta-8l\delta-45}{(27-10\delta)(l-1)}. \tag{91}$$

The right-hand side of (91) is non-positive if

$$l \leq \bar{l}_6(\delta) := \tfrac{45-29\delta}{9-8\delta}. \tag{92}$$

We conclude that for any δ the equilibrium (24) can only exist if $l \leq \bar{l}_6(\delta)$.

ii) Consider now $1/(l-1) < \alpha < (l-2)/[2(l-1)]$. In this case second-period profits are given by (10). Then firm A chooses p_A^x to maximize the profits:

$$\begin{aligned} & (l - p_A^x + p_B^x) p_A^x + \tfrac{\delta}{9}\left[(2\alpha(l-1)+1)^2 + (2l-1-\alpha(l-1))^2\right] \\ = \quad & (l - p_A^x + p_B^x) p_A^x + \tfrac{\delta}{9}\left[(2p_A^x - 2p_B^x - 1)^2 + (2l - p_A^x + p_B^x)^2\right]. \end{aligned} \tag{93}$$

Firm B chooses p_B^x to maximize the profits:

$$\begin{aligned} & (p_A^x - p_B^x - 1) p_B^x + \tfrac{\delta}{9}\left[(\alpha(l-1)-1)^2 + (l-2-2\alpha(l-1))^2\right] \\ = \quad & (p_A^x - p_B^x - 1) p_B^x + \tfrac{\delta}{9}\left[(p_A^x - p_B^x - 2)^2 + (l - 2p_A^x + 2p_B^x)^2\right]. \end{aligned} \tag{94}$$

Solving simultaneously firms' first-order conditions yields the prices:

$$p_A^{x*} = \tfrac{l(18-14\delta)+6\delta-9}{27-20\delta} \text{ and } p_B^{x*} = \tfrac{l(9-6\delta)+14\delta-18}{27-20\delta}. \tag{95}$$

Note that $p_A^{x*} \geq 0$ if

$$l \geq \tfrac{9-6\delta}{18-14\delta}. \tag{96}$$

Since for any δ the right-hand side of (96) is smaller than 1, $p_A^{x*} \geq 0$ holds for any δ and any $l \geq 4$. Note also that $p_B^{x*} \geq 0$ if

$$l \geq \tfrac{18-14\delta}{9-6\delta}. \tag{97}$$

Since for any δ the right-hand side of (97) is not larger than 2, $p_B^{x*} \geq 0$ holds for any δ and any $l \geq 4$. Using the prices (95) we can calculate the market share of firm B in the first period:

$$\alpha(p_A^{x*}, p_B^{x*}) = \tfrac{9l+12\delta-8l\delta-18}{(l-1)(27-20\delta)}.$$

The comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{x*}\right) - \tfrac{l-2}{2(l-1)} = -\tfrac{l(9-4\delta)+16\delta-18}{2(27-20\delta)(l-1)}. \tag{98}$$

The right-hand side of (98) is non-positive if

$$l \geq \tfrac{18-16\delta}{9-4\delta}. \tag{99}$$

Since the right-hand side of (99) is for any α not larger than 2, (99) holds for any δ and any $l \geq 4$. The other comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{x*}\right) - \tfrac{1}{l-1} = \tfrac{l(9-8\delta)+32\delta-45}{(27-20\delta)(l-1)}. \tag{100}$$

The right-hand side of (100) is non-negative if

$$l \geq \bar{l}_8(\delta) := \tfrac{45-32\delta}{9-8\delta}.$$

We conclude that for any δ, equilibrium (95) can only exist if $l \geq \bar{l}_8(\delta)$.

iii) Consider finally $\alpha \geq (l-2)/[2(l-1)]$. The profits of the second period are given by (9). Firm A chooses p_A^x to maximize the profits (26). Firm B chooses p_B^x to maximize the profits (27). Solving simultaneously firms' first-order conditions yields the equilibrium prices (28). These prices yield the market share of firm B in the first period (29). As follows from (30), for any δ and any $l \geq 4$ it holds that $\alpha\left(p_A^{x*}, p_B^{x*}\right) \leq 1$. The other comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{x*}\right) - \tfrac{l-2}{2(l-1)} = -\tfrac{-l(10\delta-9)+16\delta-18}{2(8\delta+27)(l-1)}. \tag{101}$$

If $\delta < 0.9$, then the right-hand side of (101) is non-negative if

$$l \leq \bar{l}_7(\delta) := \tfrac{18-16\delta}{9-10\delta}. \tag{102}$$

If $\delta > 0.9$, then the opposite to (102) inequality should hold. Finally, if $\delta = 0.9$, then the right-hand side of (101) is positive for any l. Note now that $\bar{l}_7(0.75) = 4$ and for any $\delta < 0.75$ it holds that $\bar{l}_7(\delta) < 4$, such that there is no $l \geq 4$, which satisfies (102) in that case. Note also that if $\delta > 0.9$, then $\bar{l}_7(\delta) < 0$. We conclude that the equilibrium with $\alpha \geq (l-2)/[2(l-1)]$

does not exist for any l if $\delta < 0.75$, can exist for $l \leq \bar{l}_7(\delta)$ if $0.75 \leq \delta < 0.9$ and can exist for any l if $\delta \geq 0.9$.

In the following we analyze in turn every candidate equilibrium.

6.a) Consider first the candidate equilibrium (28).

i) The incentives of firm A.

Deviation on $1/(l-1) \leq \alpha(\cdot) \leq (l-2)/[2(l-1)]$**.** If firm A deviates, it realizes the profit (93). Keeping p_B^x at p_B^{x*} in (28) and taking the derivative of (93) with respect to p_A^x yields the deviation price of firm A:

$$p_A^{xdev}(p_B^{x*}) = \frac{-l\left(-60\delta^3+360\delta^2+189\delta-972\right)-160\delta^3+228\delta^2+54\delta-486}{-240\delta^2-378\delta+1458}. \tag{103}$$

This price is non-negative if

$$l \geq \frac{54\delta+228\delta^2-160\delta^3-486}{189\delta+360\delta^2-60\delta^3-972}. \tag{104}$$

Since the right-hand side of (104) is for any δ smaller than 1, then (104) holds for any l. Using the price (103) we can calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{xdev}(p_B^{x*}), p_B^{x*}\right) = -\frac{99l\delta-72\delta-162l+14l\delta^2+324}{(l-1)\left(-80\delta^2-126\delta+486\right)}.$$

The comparison shows that

$$\alpha\left(p_A^{xdev}(p_B^{x*}), p_B^{x*}\right) - \frac{l-2}{2(l-1)} = \frac{81l+54\delta+36l\delta+80\delta^2-26l\delta^2-162}{2(l-1)\left(40\delta^2+63\delta-243\right)}. \tag{105}$$

The right-hand side of (105) is non-positive if

$$l \geq -\frac{54\delta+80\delta^2-162}{36\delta-26\delta^2+81}. \tag{106}$$

Since the right-hand side of (106) is for any δ not larger than 2, then (106) holds for any δ and any $l \geq 2$. The other comparison shows that

$$\alpha\left(p_A^{xdev}(p_B^{x*}), p_B^{x*}\right) - \frac{1}{l-1} = \frac{162l+198\delta-99l\delta+80\delta^2-14l\delta^2-810}{2(l-1)\left(-40\delta^2-63\delta+243\right)}. \tag{107}$$

The right-hand side of (107) is non-negative if

$$l \geq \frac{198\delta+80\delta^2-810}{99\delta+14\delta^2-162}. \tag{108}$$

We first consider the case (108), such that the optimal deviation price of firm A is given by (103). Then the difference between the equilibrium and the deviation profits is equal to

$$\frac{\delta\left[l^2\left(-3176\delta^3+6372\delta^2+8991\delta-20412\right)-l\left(-12160\delta^3+19152\delta^2+430092\delta-55404\right)-12800\delta^3+5760\delta^2+356640\delta-29160\right]}{36(9-5\delta)(8\delta+27)^2}. \tag{109}$$

The function in the brackets of the numerator of (109) is quadratic in l, opens downwards for any δ and has two roots, both of which are not larger than 2. Hence, for any δ and any $l \geq 2$, it takes non-positive values. We conclude that for any δ and any l, which satisfy (108), firm A has an incentive to deviate. Consider now the other case with

$$l < \frac{198\delta+80\delta^2-810}{99\delta+14\delta^2-162}. \tag{110}$$

Then the deviation price of firm A follows from $\alpha\left(p_A^{xdev}\left(p_B^{x*}\right), p_B^{x*}\right) = 1/\left(l-1\right)$ and is given by

$$p_A^{xdev}\left(p_B^{x*}\right) = \frac{l\left(-6\delta^2+21\delta+27\right)+16\delta^2+30\delta+108}{24\delta+81}, \tag{111}$$

which is positive for any δ and any l. Using the price (111), we can calculate the difference between the equilibrium and the deviation profits of firm A:

$$\frac{-l^2\left(-149\delta^3+189\delta^2+1053\delta-729\right)+l\left(-496\delta^3+1134\delta^2+3726\delta-7290\right)+320\delta^3-1296\delta^2+405\delta+18225}{9(8\delta+27)^2}. \tag{112}$$

Remember that the equilibrium (93) can exist only if $\delta \geq 0.75$. For any $\delta \geq 0.75$, the quadratic function in the numerator of (112) opens downwards and has two roots:

$$\begin{aligned} l_9\left(\delta\right) &= \frac{-\left(-496\delta^3+1134\delta^2+3726\delta-7290\right)+6(8\delta+27)\sqrt{3\delta\left(8\delta^3-70\delta^2+51\delta+270\right)}}{-2\left(-149\delta^3+189\delta^2+1053\delta-729\right)}, \\ l_{10}\left(\delta\right) &= \frac{-\left(-496\delta^3+1134\delta^2+3726\delta-7290\right)-6(8\delta+27)\sqrt{3\delta\left(8\delta^3-70\delta^2+51\delta+270\right)}}{-2\left(-149\delta^3+189\delta^2+1053\delta-729\right)}. \end{aligned}$$

Note that $l_9\left(\delta\right) < 0$ for any $\delta \geq 0.75$. For any $\delta \geq 0.75$ it also holds that $l_{10}\left(\delta\right) \geq 4$ and $l_{10}\left(\delta\right)$ satisfies (110). Finally, for any $0.75 \leq \delta < 0.9$ it holds that $l_{10}\left(\delta\right) \leq \bar{l}_7\left(\delta\right)$. We conclude that for any $\delta \geq 0.75$, firm A does not have an incentive to deviate if $4 \leq l \leq l_{10}(\delta)$.

Deviation on $\alpha(\cdot) \leq 1/(l-1)$. If firm A deviates, then its profit is (20). Keeping p_B^x at p_B^{x*} in (28) and taking the derivative of (20) with respect to p_A^x yields the price:

$$p_A^{xdev}(p_B^{x*}) = \frac{972l+675\delta+27l\delta+396\delta^2-32\delta^3-192l\delta^2+12l\delta^3-486}{-48\delta^2+270\delta+1458}. \tag{113}$$

This price is non-negative if

$$l \geq -\frac{675\delta+396\delta^2-32\delta^3-486}{27\delta-192\delta^2+12\delta^3+972}. \tag{114}$$

Note that for any δ, the right-hand side of (114) is smaller than 1, such that (114) holds for any δ and any l. Using the price (113) we calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{xdev}(p_B^{x*}), p_B^{x*}\right) = \frac{-l\left(14\delta^2+99\delta-162\right)+40\delta^2+207\delta-324}{\left(-16\delta^2+90\delta+486\right)(l-1)}. \tag{115}$$

The numerator in (115) is non-negative if

$$l \geq \frac{207\delta+40\delta^2-324}{99\delta+14\delta^2-162}. \tag{116}$$

The right-hand side of (116) is for any δ not larger than 2, such that (116) holds for any δ and any $l \geq 2$. The denominator of (115) is negative for any δ and any l. We conclude that $\alpha\left(p_A^{xdev}(p_B^{x*}), p_B^{x*}\right) \geq 0$ holds for any δ and any $l \geq 2$. The other comparison shows that

$$\alpha\left(p_A^{xdev}(p_B^{x*}), p_B^{x*}\right) - \frac{1}{l-1} = \frac{162l+117\delta-99l\delta+56\delta^2-14l\delta^2-810}{2(l-1)\left(-8\delta^2+45\delta+243\right)}. \tag{117}$$

The numerator in (117) is non-positive if

$$l \leq \frac{117\delta+56\delta^2-810}{99\delta+14\delta^2-162}. \tag{118}$$

In the following we will restrict attention to the case (118) only, because it includes $4 \leq l \leq l_{10}(\delta)$, where firm A does not have an incentive to deviate, as we showed above. Under (118), the optimal deviation price of firm A is given by (113). Using (113), we calculate the difference between the equilibrium and the deviation profits of firm A:

$$\frac{\delta\left[-l^2\left(264\delta^3-1116\delta^2+2619\delta+2916\right)+l\left(1184\delta^3-2676\delta^2+2754\delta-11\,664\right)-1472\delta^3+1200\delta^2+9585\delta+43740\right]}{12(9-\delta)(8\delta+27)^2}. \tag{119}$$

The function in the brackets in the numerator of (119) is quadratic in l, opens downwards and has two roots both of which are smaller than 4 for any δ. It then follows that for any δ and any $4 \leq l \leq l_{10}(\delta)$, the term in (119) is non-positive, such that firm A has an incentive to deviate.

Conclusion from 6.a) We conclude that for any $l \geq 4$ the equilibrium (28) does not exist.

6.b) Consider now the candidate equilibrium (24).

Deviation on $1/(l-1) \leq \alpha(\cdot) \leq (l-2)/[2(l-1)]$**.**

i) The incentives of firm A. If firm A deviates, then its profit is (93). Keeping p_B^x at p_B^{x*} in (*24*) and taking the derivative of (93) with respect to p_A^x yields the deviation price of firm A:

$$p_A^{xdev}(p_B^{x*}) = \frac{972l+513\delta-972l\delta-426\delta^2+240\delta^3+312l\delta^2-80l\delta^3-486}{300\delta^2-1350\delta+1458}. \tag{120}$$

The right-hand side of (120) is non-negative if

$$l \geq \frac{513\delta-426\delta^2+240\delta^3-486}{972\delta-312\delta^2+80\delta^3-972}. \tag{121}$$

The right-hand side of (121) is for any δ smaller than 1, such that (121) holds for any δ and any l. Using (120) we calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{xdev}(p_B^{x*}), p_B^{x*}\right) = \frac{162l+243\delta-162l\delta+12\delta^2+16l\delta^2-324}{(l-1)\left(100\delta^2-450\delta+486\right)}.$$

The comparison shows that

$$\alpha\left(p_A^{xdev}(p_B^{x*}), p_B^{x*}\right) - \frac{l-2}{2(l-1)} = -\frac{81l+207\delta-63l\delta-112\delta^2+34l\delta^2-162}{2(l-1)\left(50\delta^2-225\delta+243\right)}. \tag{122}$$

The right-hand side of (122) is non-positive if

$$l \geq \frac{-207\delta+112\delta^2+162}{-63\delta+34\delta^2+81}. \tag{123}$$

The right-hand side of (123) is for any δ not larger than 2, such that (123) holds for any δ and any $l \geq 2$. The other comparison shows that

$$\alpha\left(p_A^{xdev}(p_B^{x*}), p_B^{x*}\right) - \frac{1}{l-1} = \frac{162l+693\delta-162l\delta-88\delta^2+16l\delta^2-810}{2(l-1)\left(50\delta^2-225\delta+243\right)}. \tag{124}$$

The right-hand side of (124) is non-negative if

$$l \geq \frac{-693\delta+88\delta^2+810}{-162\delta+16\delta^2+162}. \tag{125}$$

We showed above that the equilibrium (24) can only exist if $l \leq \bar{l}_6(\delta)$, with $\bar{l}_6(\delta)$ being defined in (92). The comparison shows that

$$\bar{l}_6(\delta) - \frac{-693\delta+88\delta^2+810}{-162\delta+16\delta^2+162} = \frac{3(27-10\delta)\delta}{2\left(8\delta^2-81\delta+81\right)} \geq 0 \text{ for any } \delta.$$

Consider first

$$l \leq \frac{-693\delta+88\delta^2+810}{-162\delta+16\delta^2+162}, \tag{126}$$

in which case the optimal deviation price of firm A follows from $\alpha\left(p_A^{xdev}\left(p_B^{x*}\right), p_B^{x*}\right) = 1/(l-1)$. As the profits of firm A are continuous, we conclude that firm A does not have an incentive to deviate when (126) holds.

We consider now the other case and assume that (125) holds, such that (120) is the optimal deviation price of firm A. Using this price we calculate the difference between the equilibrium and the deviation profits of firm A:

$$\frac{\delta\left[l^2\left(1024\delta^3-11520\delta^2+22032\delta-11664\right)-l\left(6464\delta^3-74376\delta^2+155844\delta-90396\right)+9076\delta^3-111888\delta^2+253935\delta-160380\right]}{36(9-5\delta)(10\delta-27)^2}. \tag{127}$$

The function in the brackets in the numerator of (127) is quadratic in l, opens downwards for any δ and has two roots, one of which does not fulfill (125). The other root does and is given by

$$\bar{l}_9(\delta) := \frac{\left(6464\delta^3-74376\delta^2+155844\delta-90396\right)-12\left(80\delta^2-306\delta+243\right)\sqrt{5\delta^2-54\delta+81}}{2\left(1024\delta^3-11520\delta^2+22032\delta-11664\right)}.$$

For any δ it holds that $\bar{l}_9(\delta) \leq \bar{l}_6(\delta)$. We conclude then that firm A deviates for any δ if $\bar{l}_9(\delta) < l \leq \bar{l}_6(\delta)$ and does not deviate if

$$\frac{-693\delta+88\delta^2+810}{-162\delta+16\delta^2+162} < l \leq \bar{l}_9(\delta).$$

Conclusion from i). We conclude that for any δ firm A deviates if $\bar{l}_9(\delta) < l \leq \bar{l}_6(\delta)$ and does not deviate if $4 \leq l \leq \bar{l}_9(\delta)$.

ii) The incentives of firm B. If firm B deviates, then its profit is (94). Keeping p_A^x at p_A^{x*} in

(24) and taking the derivative of (94) with respect to p_B^x yields the deviation price of firm B:

$$p_B^{xdev}\left(p_A^{x*}\right) = \frac{-l\left(40\delta^3-156\delta^2+270\delta-243\right)+120\delta^3-468\delta^2+702\delta-486}{150\delta^2-675\delta+729}. \tag{128}$$

The right-hand side of (128) is non-negative if

$$l \geq \frac{702\delta-468\delta^2+120\delta^3-486}{270\delta-156\delta^2+40\delta^3-243}. \tag{129}$$

For any δ, the right-hand side of (129) is not larger than 2, such that (129) holds for any δ and any $l \geq 2$. Using (128) we calculate the share of firm B in the first period:

$$\alpha\left(p_A^{x*}, p_B^{xdev}\left(p_A^{x*}\right)\right) = \frac{81l+216\delta-108l\delta-66\delta^2+32l\delta^2-162}{(l-1)\left(50\delta^2-225\delta+243\right)}.$$

The comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{xdev}\left(p_A^{x*}\right)\right) - \frac{l-2}{2(l-1)} = -\frac{-l\left(14\delta^2+9\delta-81\right)+32\delta^2+18\delta-162}{2(l-1)\left(50\delta^2-225\delta+243\right)}. \tag{130}$$

The right-hand side of (130) is non-positive if

$$l \geq \frac{18\delta+32\delta^2-162}{9\delta+14\delta^2-81}. \tag{131}$$

For any δ the right-hand side of (131) is not larger than 2, such that (131) holds for any δ and any $l \geq 2$. The other comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{xdev}\left(p_A^{x*}\right)\right) - \frac{1}{l-1} = \frac{(9-4\delta)(9l+29\delta-8l\delta-45)}{(l-1)\left(50\delta^2-225\delta+243\right)}. \tag{132}$$

The right-hand side of (132) is non-negative if

$$l \geq \bar{l}_6\left(\delta\right) = \frac{45-29\delta}{9-8\delta}. \tag{133}$$

We showed above that the equilibrium (24) can exist only if $l \leq \bar{l}_6\left(\delta\right)$. It follows then from (133) that the optimal deviation price of firm B is given by $\alpha\left(p_A^{x*}, p_B^{xdev}\left(p_A^{x*}\right)\right) = 1/\left(l-1\right)$. From the fact that the profits of firm B are continuous, we conclude then that firm B does not have an incentive to deviate.

Conclusion from ii). We conclude that for any δ and $l \geq 4$ firm B does not have an incentive to deviate.

Deviation on $\alpha(\cdot) \geq (l-2)/[2(l-1)]$.

i) The incentives of firm A. If firm A deviates, then its profit is (26). Keeping p_B^x at p_B^{x*} in (24) and taking the derivative of (26) with respect to p_A^x yields the optimal deviation price of firm A:

$$p_A^{xdev}\left(p_B^{x*}\right) = -\frac{-l\left(80\delta^3-318\delta^2+621\delta+972\right)+240\delta^3-234\delta^2+567\delta+486}{-300\delta^2+270\delta+1458}. \tag{134}$$

The right-hand side of (134) is non-negative if

$$l \geq \frac{567\delta-234\delta^2+240\delta^3+486}{621\delta-318\delta^2+80\delta^3+972}. \tag{135}$$

The right-hand side of (135) is for any δ smaller than 1, such that (135) holds for any δ and any l. Using (134) we calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{xdev}\left(p_B^{x*}\right), p_B^{x*}\right) = \frac{297\delta-162l-189l\delta-212\delta^2+114l\delta^2+324}{(l-1)\left(100\delta^2-90\delta-486\right)}.$$

The comparison shows that

$$\alpha\left(p_A^{xdev}\left(p_B^{x*}\right), p_B^{x*}\right) - 1 = -\frac{l\left(14\delta^2-99\delta+324\right)-112\delta^2+207\delta-162}{2(l-1)\left(-50\delta^2+45\delta+243\right)}. \tag{136}$$

The right-hand side of (136) is non-positive if

$$l \geq \frac{-207\delta+112\delta^2+162}{-99\delta+14\delta^2+324}. \tag{137}$$

The right-hand side of (137) is for any δ smaller than 1, such that (137) holds for any δ and any l. The other comparison shows that

$$\alpha\left(p_A^{xdev}\left(p_B^{x*}\right), p_B^{x*}\right) - \frac{l-2}{2(l-1)} = -\frac{l\left(64\delta^2-144\delta+81\right)-112\delta^2+207\delta-162}{2(l-1)\left(-50\delta^2+45\delta+243\right)}. \tag{138}$$

The right-hand of (138) is non-negative if

$$l \leq l_{11}(\delta) := \frac{-207\delta+112\delta^2+162}{-144\delta+64\delta^2+81}. \tag{139}$$

Consider first $\max\{4, l_{11}(\delta)\} \leq l \leq \bar{l}_9(\delta)$, in which case (139) does not hold and the optimal deviation price of firm A follows from $\alpha\left(p_A^{xdev}\left(p_B^{x*}\right), p_B^{x*}\right) = (l-2)/[2(l-1)]$ and is given by

$$p_A^{xdev}\left(p_B^{x*}\right) = \frac{l\left(16\delta^2-54\delta+135\right)-48\delta^2+66\delta-108}{162-60\delta}. \tag{140}$$

The numerator of (140) is non-negative if

$$l \geq \frac{-66\delta+48\delta^2+108}{-54\delta+16\delta^2+135}. \tag{141}$$

The right-hand side of (141) is for any δ smaller than 1, such that (141) holds for any δ and any l. Using (140) we calculate the difference between the equilibrium and the deviation profits of firm A:

$$-\frac{l^2\left(436\delta^3-2376\delta^2+2025\delta-729\right)-l\left(2816\delta^3-15\,444\delta^2+14\,418\delta-2916\right)+4324\delta^3-23\,868\delta^2+23\,652\delta-2916}{36(10\delta-27)^2}. \tag{142}$$

The numerator of (142) is a quadratic function of l, which opens downwards for any δ and has two roots:

$$\frac{\left(2816\delta^3-15\,444\delta^2+14\,418\delta-2916\right)+6(27-10\delta)\sqrt{3\delta\left(36\delta^3-200\delta^2+195\delta+108\right)}}{2\left(436\delta^3-2376\delta^2+2025\delta-729\right)},$$
$$\frac{\left(2816\delta^3-15\,444\delta^2+14\,418\delta-2916\right)-6(27-10\delta)\sqrt{3\delta\left(36\delta^3-200\delta^2+195\delta+108\right)}}{2\left(436\delta^3-2376\delta^2+2025\delta-729\right)}.$$

Since for any δ both roots are smaller than 4, the numerator of (142) is negative for any δ and any $\max\{4, l_{11}(\delta)\} \leq l \leq \bar{l}_9(\delta)$, such that firm A does not have an incentive to deviate.

Consider now $4 \leq l \leq \min\left\{l_{11}(\delta), \bar{l}_9(\delta)\right\}$, in which case (139) holds and (134) is the optimal deviation price of firm A. Then the difference between the equilibrium and the deviation profits of firm A is

$$\frac{\delta\left[-l^2\left(2092\delta^3-8796\delta^2+6615\delta-2916\right)+l\left(9472\delta^3-36876\delta^2+10530\delta+11664\right)-11\,388\delta^3+42\,264\delta^2+5805\delta-43\,740\right]}{12(5\delta+9)(10\delta-27)^2}. \tag{143}$$

The function in the numerator of (143) is quadratic in l, opens upwards for any δ and has two roots:

$$\frac{-\left(9472\delta^3-36\,876\delta^2+10\,530\delta+11\,664\right)+4(27-10\delta)\sqrt{-\left(3485\delta^4-16\,032\delta^3+801\delta^2+42\,930\delta-55\,404\right)}}{-2\left(2092\delta^3-8796\delta^2+6615\delta-2916\right)},$$
$$\frac{-\left(9472\delta^3-36\,876\delta^2+10\,530\delta+11\,664\right)-4(27-10\delta)\sqrt{-\left(3485\delta^4-16\,032\delta^3+801\delta^2+42\,930\delta-55\,404\right)}}{-2\left(2092\delta^3-8796\delta^2+6615\delta-2916\right)},$$

both of which are for any δ smaller than 4, such that the numerator of (143) is non-negative for any δ and any $4 \leq l \leq \min\left\{l_{11}(\delta), \bar{l}_9(\delta)\right\}$ and firm A does not have an incentive to deviate.

Conclusion from i). We conclude that for any δ and $4 \leq l \leq \bar{l}_9(\delta)$ firm A does not have an incentive to deviate.

ii) The incentives of firm B. If firm B deviates, then it realizes the profit (27). Keeping p_A^x at p_A^{x*} in (24) and taking the derivative of (27) with respect to p_B^x yields the deviation price of firm B:

$$p_B^{xdev}\left(p_A^{x*}\right) = \frac{-l\left(8\delta^3-72\delta^2+216\delta-243\right)+24\delta^3-228\delta^2+594\delta-486}{30\delta^2-351\delta+729}. \tag{144}$$

The right-hand side of (144) is non-negative if

$$l \geq \frac{594\delta-228\delta^2+24\delta^3-486}{216\delta-72\delta^2+8\delta^3-243}. \tag{145}$$

The right-hand side of (145) is for any δ not larger than 2, such that (145) holds for any δ and any $l \geq 2$. Using (144) we calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{x*}, p_B^{xdev}\left(p_A^{x*}\right)\right) = \frac{81l+108\delta-54l\delta-26\delta^2+12l\delta^2-162}{(l-1)\left(10\delta^2-117\delta+243\right)}.$$

The comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{xdev}\left(p_A^{x*}\right)\right) - 1 = -\frac{-l\left(2\delta^2+63\delta-162\right)+16\delta^2+9\delta-81}{(l-1)\left(10\delta^2-117\delta+243\right)}. \tag{146}$$

The right-hand side of (146) is non-positive if

$$l \geq \frac{9\delta+16\delta^2-81}{63\delta+2\delta^2-162}. \tag{147}$$

The right-hand side of (147) is for any δ smaller than 1, such that (147) holds for any δ and any

l. The other comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{xdev}\left(p_A^{x*}\right)\right) - \frac{l-2}{2(l-1)} = -\frac{-l\left(14\delta^2+9\delta-81\right)+32\delta^2+18\delta-162}{2(l-1)\left(10\delta^2-117\delta+243\right)}. \tag{148}$$

The right-hand side of (148) is non-negative if

$$l \leq \frac{18\delta+32\delta^2-162}{9\delta+14\delta^2-81}. \tag{149}$$

The right-hand side of (149) is for any δ not larger than 2, such that (149) does not hold for any δ and any $l > 2$. Hence, $p_B^{xdev}\left(p_A^{x*}\right)$ follows from $\alpha\left(\cdot\right) = \left(l-2\right)/\left[2\left(l-1\right)\right]$ and is given by

$$p_B^{xdev}\left(p_A^{x*}\right) = \frac{l\left(16\delta^2-42\delta+27\right)-48\delta^2+120\delta-54}{162-60\delta}. \tag{150}$$

The right-hand side of (150) is non-negative if

$$l \geq \frac{-120\delta+48\delta^2+54}{-42\delta+16\delta^2+27}. \tag{151}$$

The right-hand side of (151) is for any δ not larger than 2, such that (151) holds for any δ and any $l \geq 2$. Hence, (150) is the optimal deviation price of firm B. Using (150) we calculate the difference between the equilibrium and the deviation profits of firm B:

$$-\frac{l^2\left(244\delta^3-432\delta^2+81\delta-729\right)-l\left(1664\delta^3-4104\delta^2+2268\delta-2916\right)+2896\delta^3-8964\delta^2+7128\delta-2916}{36(10\delta-27)^2}. \tag{152}$$

The numerator of (152) is a quadratic function of l, which opens downwards for any δ and has two roots:

$$\frac{1664\delta^3-4104\delta^2+2268\delta-2916-12\left(81-10\delta^2-3\delta\right)\sqrt{\delta(9-4\delta)}}{2\left(244\delta^3-432\delta^2+81\delta-729\right)}, \frac{1664\delta^3-4104\delta^2+2268\delta-2916+12\left(81-10\delta^2-3\delta\right)\sqrt{\delta(9-4\delta)}}{2\left(244\delta^3-432\delta^2+81\delta-729\right)},$$

both of which are for any δ smaller than 4. Hence, for any δ and any $4 \leq l \leq \bar{l}_9\left(\delta\right)$ the numerator of (152) is non-positive, such that firm B does not have an incentive to deviate.

Conclusion from ii). We conclude that for any δ and any $4 \leq l \leq \bar{l}_9\left(\delta\right)$ firm B does not have an incentive to deviate.

Conclusion from 6.b). We conclude that equilibrium (24) exists for any δ and any $4 \leq l \leq$

$\bar{l}_9(\delta)$.

6.c) Consider finally the candidate equilibrium (95).

Deviation on $\alpha(\cdot) \leq 1/(l-1)$.

i) The incentives of firm A. If firm A deviates, then its profit is (20). Keeping p_B^x at p_B^{x*} in (95) and taking the derivative of (20) with respect to p_A^x yields the deviation price of firm A:

$$p_A^{xdev}(p_B^{x*}) = \frac{l(92\delta^2-360\delta+324)-208\delta^2+405\delta-162}{40\delta^2-414\delta+486}. \tag{153}$$

The right-hand side of (153) is non-negative if

$$l \geq \frac{-405\delta+208\delta^2+162}{-360\delta+92\delta^2+324}. \tag{154}$$

Since the right-hand side of (154) is for any δ smaller than 1, (154) holds for any δ and l. Using (153) we calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{xdev}(p_B^{x*}), p_B^{x*}\right) = \frac{l(80\delta^2-234\delta+162)-220\delta^2+531\delta-324}{(l-1)(40\delta^2-414\delta+486)}. \tag{155}$$

The right-hand side of (155) is non-negative if

$$l \geq \frac{-531\delta+220\delta^2+324}{-234\delta+80\delta^2+162}. \tag{156}$$

For any δ the right-hand side of (156) is not larger than 2, such that (156) holds for any δ and any $l \geq 2$. The other comparison shows that

$$\alpha\left(p_A^{xdev}(p_B^{x*}), p_B^{x*}\right) - \frac{1}{l-1} = \frac{l(80\delta^2-234\delta+162)-260\delta^2+945\delta-810}{2(l-1)(20\delta^2-207\delta+243)}. \tag{157}$$

The right-hand side of (157) is non-positive if

$$l \leq l_{12}(\delta) := \frac{-945\delta+260\delta^2+810}{-234\delta+80\delta^2+162}.$$

We showed above that the equilibrium (95) can only exist if $l \geq \bar{l}_8(\delta)$. The comparison shows that

$$l_{12}(\delta) - \bar{l}_8(\delta) = \frac{3\delta(27-20\delta)}{2(40\delta^2-117\delta+81)} \geq 0 \text{ for any } \delta.$$

We next analyze two cases. Consider first $l \geq l_{12}(\delta)$, in which case the optimal deviation price of firm A follows from $\alpha\left(p_A^{xdev}\left(p_B^{x*}\right), p_B^{x*}\right) = 1/\left(l-1\right)$. As firm A's profits are continuous, we conclude that firm A does not have an incentive to deviate.

Consider next $\bar{l}_8(\delta) \leq l \leq l_{12}(\delta)$, in which case the optimal deviation price of firm A is given by (153). Then the difference between the equilibrium and the deviation profits of firm A is

$$-\frac{\delta\left[l^2\left(5120\delta^3-20\,736\delta^2+27\,216\delta-11\,664\right)-l\left(31\,360\delta^3-136\,368\delta^2+194\,076\delta-90\,396\right)+a(\delta)\right]}{36(9-\delta)(20\delta-27)^2}, \text{ with} \tag{158}$$

$$a(\delta) := 47120\delta^3 - 212616\delta^2 + 319869\delta - 160380.$$

The function in the brackets of (158) is quadratic in l, opens downwards for any δ and has two roots:

$$\frac{\left(31\,360\delta^3-136\,368\delta^2+194\,076\delta-90\,396\right)+12\left(160\delta^2-396\delta+243\right)\sqrt{5\delta^2-54\delta+81}}{2\left(5120\delta^3-20\,736\delta^2+27\,216\delta-11\,664\right)},$$

$$\bar{l}_{10}(\delta) \quad := \quad \frac{\left(31\,360\delta^3-136\,368\delta^2+194\,076\delta-90\,396\right)-12\left(160\delta^2-396\delta+243\right)\sqrt{5\delta^2-54\delta+81}}{2\left(5120\delta^3-20\,736\delta^2+27\,216\delta-11\,664\right)}.$$

The first root is for any δ smaller than $\bar{l}_8(\delta)$, while the other root for any δ fulfills $\bar{l}_8(\delta) \leq \bar{l}_{10}(\delta) < l_{12}(\delta)$. Hence, on the interval $\bar{l}_8(\delta) \leq l < l_{12}(\delta)$, firm A does not deviate if $\bar{l}_{10}(\delta) \leq l < l_{12}(\delta)$ and deviates otherwise.

Conclusion from i). Combining the results from both cases we conclude that firm A does not deviate if $l \geq \bar{l}_{10}(\delta)$ and deviates if $\bar{l}_8(\delta) \leq l < \bar{l}_{10}(\delta)$.

ii) The incentives of firm B. If firm B deviates, then its profit is given by (21). Keeping p_A^x at p_A^{x*} in (95) and taking the derivative of (21) with respect to p_B^x yields the deviation price of firm B:

$$p_B^{xdev}\left(p_A^{x*}\right) = \frac{l\left(16\delta^2-81\delta+81\right)-24\delta^2+153\delta-162}{80\delta^2-288\delta+243}. \tag{159}$$

The right-hand side of (159) is non-negative if

$$l \geq \frac{-153\delta+24\delta^2+162}{-81\delta+16\delta^2+81}. \tag{160}$$

The right-hand side of (160) is for any δ smaller than 4, such that (160) holds for any δ and any

$l \geq 4$. Using (159) we calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{x*}, p_B^{xdev}\left(p_A^{x*}\right)\right) = \frac{l\left(40\delta^2-117\delta+81\right)-80\delta^2+225\delta-162}{(l-1)\left(80\delta^2-288\delta+243\right)}. \tag{161}$$

The right-hand side of (161) is non-negative if

$$l \geq \frac{-225\delta+80\delta^2+162}{-117\delta+40\delta^2+81}. \tag{162}$$

Note that for any δ and any $l \geq \bar{l}_8(\delta)$, the inequality (162) holds. The other comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{xdev}\left(p_A^{x*}\right)\right) - \frac{1}{l-1} = \frac{(9-5\delta)(9l+32\delta-8l\delta-45)}{(l-1)\left(80\delta^2-288\delta+243\right)}. \tag{163}$$

The right-hand side of (163) is non-positive if $l \leq \bar{l}_8(\delta)$ holds. Since the equilibrium (95) can only exist if $l \geq \bar{l}_8(\delta)$, the optimal deviation price of firm B follows from $\alpha(\cdot) = 1/(l-1)$. As the profits of firm B are continuous, we conclude that firm B does not have an incentive to deviate.

Conclusion from ii). We conclude that for any δ and any $l \geq \bar{l}_8(\delta)$, firm B does not have an incentive to deviate.

Deviation on $\alpha(\cdot) \geq (l-2)/[2(l-1)]$.

i) The incentives of firm A. If firm A deviates, then its profit is (26). Keeping p_B^x at p_B^{x*} in (95) and taking the derivative of (26) with respect to p_A^x yields the deviation price of firm A:

$$p_A^{xdev}\left(p_B^{x*}\right) = \frac{-l\left(-240\delta^2+99\delta+324\right)-220\delta^2+162\delta+162}{200\delta^2+90\delta-486}, \tag{164}$$

which is non-negative if

$$l \geq \frac{162\delta-220\delta^2+162}{99\delta-240\delta^2+324}. \tag{165}$$

Since for any δ the right-hand side of (165) is smaller than 1, then (165) holds for any δ and any l. Using (164) we calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{xdev}\left(p_B^{x*}\right), p_B^{x*}\right) = -\frac{144\delta-162l-117l\delta-280\delta^2+180l\delta^2+324}{2(l-1)\left(-100\delta^2-45\delta+243\right)}.$$

The comparison shows that

$$\alpha\left(p_A^{xdev}\left(p_B^{x*}\right), p_B^{x*}\right) - 1 = \frac{l\left(20\delta^2+207\delta-324\right)+80\delta^2-234\delta+162}{2(l-1)\left(-100\delta^2-45\delta+243\right)}. \tag{166}$$

The right-hand side of (166) is non-positive if

$$l \geq -\frac{-234\delta+80\delta^2+162}{207\delta+20\delta^2-324}. \tag{167}$$

Since for any δ the right-hand side of (167) is smaller than 1, (167) holds for any δ and any l. The other comparison shows that

$$\alpha\left(p_A^{xdev}\left(p_B^{x*}\right), p_B^{x*}\right) - \frac{l-2}{2(l-1)} = \frac{(9-8\delta)(9l+10\delta-10l\delta-18)}{2(l-1)\left(100\delta^2+45\delta-243\right)}. \tag{168}$$

The right-hand side of (168) is non-negative if

$$-l\left(10\delta-9\right)+10\delta-18 \leq 0 \tag{169}$$

holds. We have to distinguish between the cases: *a)* if $\delta < 0.9$, then (169) holds for $l \leq (18-10\delta)/(9-10\delta)$ and *b)* if $\delta \geq 0.9$, then (169) holds for any l. Remember that the equilibrium (95) exists if $l \geq \bar{l}_8(\delta)$. Note next that if $\delta = 0.75$, then $\bar{l}_8(\delta) = (18-10\delta)/(9-10\delta)$. If $\delta < 0.75$, then for any $l \geq \bar{l}_8(\delta)$ (169) does not hold. Similarly, if $0.75 \leq \delta < 0.9$ and $l > (18-10\delta)/(9-10\delta)$, then (169) does not hold either, in which case the optimal deviation price of firm A follows from $\alpha(\cdot) = (l-2)/[2(l-1)]$. Hence, in these cases firm A does not have an incentive to deviate because its profits are continuous.

Consider the remaining two cases: *c)* $0.75 \leq \delta < 0.9$ and $\bar{l}_8(\delta) \leq l \leq (18-10\delta)/(9-10\delta)$, *d)* $\delta \geq 0.9$ and $l \geq \bar{l}_8(\delta)$. In these cases (169) is fulfilled and (164) is the optimal deviation price of firm A. Using (164) we calculate the difference between the equilibrium and the deviation profits of firm A:

$$-\frac{\delta\left[l^2\left(6800\delta^3-27720\delta^2+39933\delta-20412\right)-l\left(16000\delta^3-74160\delta^2+112428\delta-55404\right)+12800\delta^3-51840\delta^2+68040\delta-29160\right]}{36(5\delta+9)(20\delta-27)^2}. \tag{170}$$

The function in the brackets in (170) is quadratic in l, opens downwards and has two roots both of which are for any δ smaller than 4. Hence, the expression in (170) is non-negative for any δ

and any $l \geq 4$, which implies that firm A does not have an incentive to deviate.

Conclusion from i). We conclude that firm A does not have an incentive to deviate**.**

ii) The incentives of firm B. If firm B deviates, then its profit is (27). Keeping p_A^x at p_A^{x*} in (95) and taking the derivative of (27) with respect to p_B^x yields the deviation price of firm B:

$$p_B^{xdev}\left(p_A^{x*}\right) = \frac{l\left(14\delta^2-81\delta+81\right)-46\delta^2+180\delta-162}{20\delta^2-207\delta+243}, \tag{171}$$

which is non-negative if

$$l \geq \frac{-180\delta+46\delta^2+162}{-81\delta+14\delta^2+81}. \tag{172}$$

The right-hand side of (172) is for any δ not larger than 2, such that (172) holds for any δ and any $l \geq 2$. Using (171) we calculate the market share of firm B in the first period:

$$\alpha\left(p_A^{x*}, p_B^{xdev}\left(p_A^{x*}\right)\right) = \frac{81l+90\delta-63l\delta+20\delta^2-162}{(l-1)\left(20\delta^2-207\delta+243\right)}.$$

The comparison shows that

$$\alpha\left(p_A^{x*}, p_B^{xdev}\left(p_A^{x*}\right)\right) - \frac{l-2}{2(l-1)} = -\frac{(9-5\delta)(9l+16\delta-4l\delta-18)}{2(l-1)\left(20\delta^2-207\delta+243\right)},$$

which is non-negative if

$$l \leq \frac{18-16\delta}{9-4\delta}. \tag{173}$$

The right-hand side of (173) is for any δ not larger than 2, such that (173) does not hold for any δ and any $l > 2$. Hence, the optimal deviation price of firm B follows from $\alpha(\cdot) = (l-2)/[2(l-1)]$, which together with the fact that the profits of firm B are continuous, implies that firm B does not have an incentive to deviate.

Conclusion from ii). We conclude that firm B does not have an incentive to deviate**.**

Conclusion from 6.c) We conclude that the equilibrium (*95*) exists for any δ and any $l \geq \bar{l}_{10}(\delta)$.

In Proposition 2 we use the following new notation:

$$\begin{aligned}
h_1(\delta) &: = \bar{l}_1(\delta) = \tfrac{2+\delta}{1+\delta}, \\
h_2(\delta) &: = l_4(\delta) = \frac{24\delta^3+124\delta^2+176\delta+76+4(\delta+1)(2\delta+3)\sqrt{(\delta+1)(9-\delta)}}{2\left(4\delta^3+24\delta^2+49\delta+28\right)}, \\
h_3(\delta) &: = l_2(\delta) = \frac{-120\delta^3+276\delta^2+6552\delta+6156+36(27-10\delta)(1+\delta)\sqrt{(\delta+1)(9-\delta)}}{2\left(-100\delta^3+372\delta^2+531\delta+2268\right)}, \\
h_4(\delta) &: = \bar{l}_9(\delta) = \frac{\left(6464\delta^3-74376\delta^2+155844\delta-90396\right)-12\left(80\delta^2-306\delta+243\right)\sqrt{5\delta^2-54\delta+81}}{2\left(1024\delta^3-11520\delta^2+22032\delta-11664\right)}, \\
h_5(\delta) &: = \bar{l}_{10}(\delta) = \frac{\left(31\,360\delta^3-136\,368\delta^2+194\,076\delta-90\,396\right)-12\left(160\delta^2-396\delta+243\right)\sqrt{5\delta^2-54\delta+81}}{2\left(5120\delta^3-20\,736\delta^2+27\,216\delta-11\,664\right)}.
\end{aligned}$$

We can now summarize our results as follows. If $l \leq h_1(\delta)$, then in equilibrium in the first period firms charge prices (19) and realize profits (13) and (14). If $h_1(\delta) < l \leq h_2(\delta)$, then firms realize the same profits but charge the prices (15). If $h_3(\delta) \leq l \leq \min\{h_4(\delta), h_5(\delta)\}$, then in equilibrium firms charge prices (24) and realize profits (20) and (21). Finally, if $l \geq \max\{h_4(\delta), h_5(\delta)\}$, then in equilibrium firms charge prices (95) and realize profits (93) and (94).

Note finally that if $\delta \lesssim 0.98$, then $\min\{h_4(\delta), h_5(\delta)\} = h_4(\delta)$ and the other way around if $\delta > 0.98$. Our results show that if $h_2(\delta) < l < h_3(\delta)$ and $h_4(\delta) < l < h_5(\delta)$, then no equilibrium in pure strategies in the first period exists. If $h_5(\delta) < l < h_4(\delta)$, then two equilibria exist, where in the first period firms charge prices (24) or (95). *Q.E.D.*

Proof of Corollary 1. We use the results on the equilibrium market share of firm B derived in the proof of Proposition 2. If $l \leq h_1(\delta)$, then $\alpha^*(\delta, l) = 0$. If $h_1(\delta) < l \leq h_2(\delta)$, then $\alpha^*(\delta, l) = [l(1+\delta) - 2 - \delta] / [(2\delta+3)(l-1)]$. If $h_3(\delta) \leq l \leq \min\{h_4(\delta), h_5(\delta)\}$, then $\alpha^*(\delta, l) = [l(9-8\delta) + 19\delta - 18] / [(l-1)(27-10\delta)]$. Finally, if $l \geq \max\{h_4(\delta), h_5(\delta)\}$, then $\alpha^*(\delta, l) = [l(9-8\delta) + 6(2\delta-3)] / [(l-1)(27-20\delta)]$. Taking the derivatives of these market shares yields the results stated in the corollary. *Q.E.D.*

Proof of Corollary 2. To prove the corollary we will use the results stated in the proof of Proposition 2. In that proof we derived the equilibrium adjusted profits (divided by $\underline{t}(1-2x)$ and multiplied by $(l-1)$) of each firm on a given location on firm A's turf. Note that these profits do not depend on the location and firms are symmetric. Hence, to analyze how the ability to collect additional flexibility data influences profits it is sufficient to compare the sum of both firms' adjusted equilibrium profits on some location. Precisely, for the case without additional customer data we evaluate the respective profits at $\delta = 0$ and then multiply them with $1 + \delta$ to get the discounted sum of profits over two periods. In the following we derive first the profits without additional customer data.

Consider $l \leq 2$. Evaluating the sum of (13) and (14) at (19) and $\delta = 0$ and then multiplying with $1 + \delta$ yields

$$(1 + \delta)(l - 1). \tag{174}$$

Consider $2 < l \leq 5$. Evaluating the sum of (20) and (21) at (24) and $\delta = 0$ and then multiplying with $1 + \delta$ yields

$$\frac{(1+\delta)\left(5l^2-8l+5\right)}{9}. \tag{175}$$

Consider finally $l > 5$. Evaluating the sum of (93) and (94) at (95) and $\delta = 0$ and then multiplying with $1 + \delta$ yields again (175).

We now compare a firms' equilibrium discounted profits with and without the additional customer data. If $l \leq h_1(\delta)$, then $\alpha^*(\delta, l) = 0$, such that no additional data is revealed in equilibrium and profits do not depend on firms' ability to collect it. Consider $h_1(\delta) < l \leq 2$. The difference between the sum of (13) and (14) evaluated at (15) and (174) is equal to

$$\frac{l^2\left(\delta^3+6\delta^2+10\delta+5\right)-l\left(2\delta^3+14\delta^2+28\delta+17\right)+\delta^3+8\delta^2+19\delta+14}{(2\delta+3)^2}. \tag{176}$$

The nominator of (176) is a quadratic function with respect to l, which opens upwards (for any δ) and has two roots: $r_1(\delta) := \left(\delta^2 + 6\delta + 7\right) / \left(\delta^2 + 5\delta + 5\right)$ and $r_2(\delta) := h_1(\delta) = (2 + \delta) / (1 + \delta)$. Note that for any δ, $r_1(\delta) < h_1(\delta)$ holds, such that for any $l > h_1(\delta)$ the nominator of (176) is positive and firms are better-off when they can obtain additional customer data.

Consider $2 < l \leq h_2(\delta)$. The difference between the sum of (13) and (14) evaluated at (15) and (175) is equal to

$$-\frac{\delta\left[l^2\left(11\delta^2+26\delta+15\right)-l\left(50\delta^2+146\delta+105\right)+47\delta^2+152\delta+123\right]}{9(2\delta+3)^2}. \tag{177}$$

The function in the brackets in (177) is quadratic in l, opens upwards (for any δ) and has two roots:

$$r_3(\delta) = \frac{(50\delta^2+146\delta+105)-3(2\delta+3)\sqrt{3(2\delta+3)(2\delta+5)}}{2(11\delta^2+26\delta+15)},$$
$$r_4(\delta) = \frac{(50\delta^2+146\delta+105)+3(2\delta+3)\sqrt{3(2\delta+3)(2\delta+5)}}{2(11\delta^2+26\delta+15)}.$$

For any δ it holds $r_3(\delta) < 2$ and $r_4(\delta) > h_2(\delta)$, such that for any $\delta > 0$ and $2 < l \leq h_2(\delta)$, the function in the brackets in (177) is negative and firms are better off when they can collect additional customer data.

Consider $h_3(\delta) \leq l \leq \min\{h_4(\delta), h_5(\delta)\}$. The difference between the sum of (20) and (21) evaluated at (24) and (175) is equal to

$$-\frac{\delta\left[l^2(560\delta^2-2380\delta+2079)-l(2480\delta^2-10864\delta+9855)+2525\delta^2-11452\delta+10665\right]}{9(10\delta-27)^2}. \tag{178}$$

The function in the brackets in (178) is quadratic in l, opens upwards (for any δ) and has two roots:

$$r_5(\delta) = \frac{(2480\delta^2-10\,864\delta+9855)-(27-10\delta)\sqrt{3(1648\delta^2-5084\delta+3855)}}{2(560\delta^2-2380\delta+2079)},$$
$$h_6(\delta) := r_6(\delta) = \frac{(2480\delta^2-10\,864\delta+9855)+(27-10\delta)\sqrt{3(1648\delta^2-5084\delta+3855)}}{2(560\delta^2-2380\delta+2079)}.$$

For any δ it holds that $r_5(\delta) < h_3(\delta)$ and $h_3(\delta) < r_6(\delta) < \min\{h_4(\delta), h_5(\delta)\}$. Hence, for any $\delta > 0$ firms are (weakly) better off with the ability to collect additional customer data if $h_3(\delta) \leq l \leq h_6(\delta)$ and are worse off otherwise. Note that $\partial h_6(\delta)/\partial\delta > 0$ for any δ.

Consider finally $l \geq \max\{h_4(\delta), h_5(\delta)\}$. The difference between the sum of (93) and (94) evaluated at (95) and (175) is equal to

$$-\frac{8\delta\left[l^2(80\delta^2-209\delta+135)-l(240\delta^2-662\delta+459)+80\delta^2-209\delta+135\right]}{9(20\delta-27)^2}. \tag{179}$$

The function in the brackets of (179) is quadratic in l, opens upwards (for any δ) and has two roots:

$$r_7(\delta) = \frac{(240\delta^2-662\delta+459)-(27-20\delta)\sqrt{80\delta^2-244\delta+189}}{2(80\delta^2-209\delta+135)} \text{ and}$$
$$r_8(\delta) = \frac{(240\delta^2-662\delta+459)+(27-20\delta)\sqrt{80\delta^2-244\delta+189}}{2(80\delta^2-209\delta+135)}.$$

Note that for any δ it holds that $r_n(\delta) < \min\{h_4(\delta), h_5(\delta)\}$, with $n = 7, 8$. Hence, for any $\delta > 0$ and any $l \geq \min\{h_4(\delta), h_5(\delta)\}$, the nominator of (179) is positive and firms are worse off with the ability to collect additional customer data.

We finally summarize our results (for $\delta > 0$). Note that $h_1(1) = 1.5$. Hence, for any $l \leq 1.5$, irrespective of δ, profits do not depend on whether firms can collect additional customer data. For any $1.5 < l < 2$ profits are larger when firms can collect data if $\delta > h_1^{-1}(l)$ and are same otherwise. Remember next that for any δ it holds that $\partial h_n(\delta)/\partial\delta > 0$, with $n = 2, ..., 6$. It also holds that $h_3(1) \approx 2.89 < h_6(0) \approx 3.07$. Hence, for any $2 \leq l \leq 3.07$ firms are better off with the ability to collect additional data irrespective of the discount factor. Note next that $h_6(1) \approx 4$. Hence, for $3.07 < l \leq 4$ firms are better of if $\delta > h_6^{-1}(l)$ and are (weakly) worse off otherwise. Finally, if $l > 4$, then firms are worse off with the ability to collect additional data irrespective of the discount factor. *Q.E.D.*

Proof of Corollary 3. For any l and δ we first calculate the equilibrium discounted social welfare over two periods and then subtract equilibrium profits to dirive consumer surplus. We then analyze how social welfare and consumer surplus change when firms are able to recognize consumers.

Part 1. i) If $l \leq h_1(\delta)$, then each firm serves all consumers on any location on its turf in both periods, such that social welfare is

$$SW_1^{1+2}(l,\delta) = v(1+\delta) - \frac{2(1+\delta)}{1-\underline{t}} \int_{\underline{t}}^{1} \int_{0}^{1/2} (tx)\, dx dt = v(1+\delta) - \frac{(1+\delta)(l+1)\underline{t}}{8}.$$

The discounted profits of a firm over two periods are given by the sum of (13) and (14) evaluated at (19), multiplied by $\underline{t}(1-2x)/(l-1))$ and integrated over $x \in [0, 1/2]$, which yields the discounted profits of both firms over two periods: $\Pi_1^{1+2}(l,\delta) = (1+\delta)\underline{t}/2$. The difference between $SW_1^{1+2}(l,\delta)$ and $\Pi_1^{1+2}(l,\delta)$ yields consumer surplus: $CS_1^{1+2}(l,\delta) = v(1+\delta) - (1+\delta)(l+5)\underline{t}/8$.

ii) If $h_1(\delta) < l \leq h_2(\delta)$, then the discounted profits of a firm over two periods are given by the sum of (13) and (14) evaluated at (15), multiplied by $\underline{t}(1-2x)/(l-1)$ and integrated over $x \in [0, 1/2]$, which yields the discounted profits of both firms over two periods:

$$\Pi_2^{1+2}(l,\delta) = \frac{\underline{t}\left[l^2\left(\delta^3+6\delta^2+10\delta+5\right) - l\left(-2\delta^3-2\delta^2+7\delta+8\right) - 3\delta^3 - 8\delta^2 - 2\delta + 5\right]}{2(2\delta+3)^2(l-1)}.$$

We calculate now social welfare. On each location on its turf a firm serves consumers with

$t \geq t^{\alpha}(\cdot) = \underline{t}(1+\delta)(l+1)/(2\delta+3)$ (we used (11) to compute $t^{\alpha}(\cdot)$) in the first period and all consumers in the second period, which yields the discounted social welfare over two periods, $SW_2^{1+2}(l,\delta) =$

$$\begin{aligned} & v(1+\delta) - \frac{2}{1-\underline{t}} \int_{\frac{\underline{t}(1+\delta)(l+1)}{2\delta+3}}^{1} \int_0^{1/2} (tx)\, dxdt - \frac{2}{1-\underline{t}} \int_{\underline{t}}^{\frac{\underline{t}(1+\delta)(l+1)}{2\delta+3}} \int_0^{1/2} [t(1-x)]\, dxdt - \frac{2\delta}{1-\underline{t}} \int_{\underline{t}}^{1} \int_0^{1/2} (tx)\, dxdt \\ = \; & v(1+\delta) - \frac{\underline{t}\left[l^2\left(4\delta^3+18\delta^2+25\delta+11\right)+l\left(4\delta^2+8\delta+4\right)-4\delta^3-22\delta^2-41\delta-25\right]}{8(l-1)(2\delta+3)^2}. \end{aligned}$$

Subtracting $\Pi_2^{1+2}(l,\delta)$ from $SW_2^{1+2}(l,\delta)$ we get the discounted consumer surplus over two periods:

$$CS_2^{1+2}(l,\delta) = v(1+\delta) + \frac{\underline{t}\left[-l^2\left(8\delta^3+42\delta^2+65\delta+31\right)+l\left(-8\delta^3-12\delta^2+20\delta+28\right)+16\delta^3+54\delta^2+49\delta+5\right]}{8(2\delta+3)^2(l-1)}.$$

iii) If $h_3(\delta) \leq l \leq \min\{h_4(\delta), h_5(\delta)\}$, then the discounted profits of a firm over two periods are given by the sum of (20) and (21) evaluated at the prices (24) multiplied by $\underline{t}(1-2x)/(l-1)$ and integrated over $x \in [0, 1/2]$, which yields the discounted profits of both firms over two periods:

$$\Pi_3^{1+2}(l,\delta) = -\frac{\underline{t}\left[l^2\left(20\delta^3-60\delta^2+378\delta-1215\right)-l\left(560\delta^3-2448\delta^2+2781\delta-1944\right)+675\delta^3-3084\delta^2+3240\delta-1215\right]}{6(10\delta-27)^2(l-1)}.$$

We calculate now social welfare. On each location on its turf a firm serves in the first period consumers with $t \geq t^{\alpha}(\cdot) = \underline{t}[l(9-8\delta)+9(1+\delta)]/(27-10\delta)$ and in the second period all consumers on segment α and those with $t \geq \underline{t}[l+1+\alpha(l-1)]/3 = 3(4l+\delta-2l\delta+1)\underline{t}/(27-10\delta)$ on segment $1-\alpha$ (we used (12) to compute α), which yields the discounted social welfare over two periods, $SW_3^{1+2}(l,\delta) =$

$$\begin{aligned} &= v(1+\delta) - \tfrac{2}{1-\underline{t}} \int_{\frac{\underline{t}[l(9-8\delta)+9(1+\delta)]}{27-10\delta}}^{1} \int_0^{1/2} (tx)\,dxdt - \tfrac{2}{1-\underline{t}} \int_{\underline{t}}^{\frac{\underline{t}[l(9-8\delta)+9(1+\delta)]}{27-10\delta}} \int_0^{1/2} [t(1-x)]\,dxdt \\ &\quad - \tfrac{2\delta}{1-\underline{t}} \int_{\underline{t}}^{\frac{\underline{t}[l(9-8\delta)+9(1+\delta)]}{27-10\delta}} \int_0^{1/2} (tx)\,dxdt - \tfrac{2\delta}{1-\underline{t}} \int_{\frac{\underline{t}[l(9-8\delta)+9(1+\delta)]}{27-10\delta}}^{\frac{3(4l+\delta-2l\delta+1)\underline{t}}{27-10\delta}} \int_0^{1/2} [t(1-x)]\,dxdt \\ &\quad - \tfrac{2\delta}{1-\underline{t}} \int_{\frac{3(4l+\delta-2l\delta+1)\underline{t}}{27-10\delta}}^{1} \int_0^{1/2} (tx)\,dxdt \\ &= v(1+\delta) - \tfrac{\underline{t}\left[l^2\left(44\delta^3-312\delta^2+27\delta+891\right)-l\left(-216\delta^3+252\delta^2+144\delta-324\right)-244\delta^3+114\delta^2+1071\delta-2025\right]}{8(l-1)(10\delta-27)^2}. \end{aligned}$$

Subtracting $\Pi_3^{1+2}(l,\delta)$ from $SW_3^{1+2}(l,\delta)$ we get the discounted consumer surplus over two periods, $CS_3^{1+2}(l,\delta) =$

$$= v(1+\delta) + \tfrac{\underline{t}\left[l^2\left(-52\delta^3+696\delta^2+1431\delta-7533\right)-l\left(2888\delta^3-10\,548\delta^2+10\,692\delta-6804\right)+3432\delta^3-12\,678\delta^2+9747\delta+1215\right]}{24(10\delta-27)^2(l-1)}.$$

iv) If $l > \max\{h_4(\delta), h_5(\delta)\}$, then the discounted profits of a firm are given by the sum of (93) and (94) evaluated at (95), multiplied by $\underline{t}(1-2x)/(l-1)$ and integrated over $x \in [0, 1/2]$, which yields the discounted profits of both firms over two periods:

$$\Pi_4^{1+2}(l,\delta) = -\tfrac{\underline{t}\left[l^2\left(-1360\delta^3+1728\delta^2+2835\delta-3645\right)-l\left(-1280\delta^3+144\delta^2+6480\delta-5832\right)-1360\delta^3+1728\delta^2+2835\delta-3645\right]}{18(20\delta-27)^2(l-1)}.$$

We now compute social welfare. On each location on its turf a firm serves in the first period consumers with $t \geq t^{\alpha}(\cdot) = \underline{t}[(9-8\delta)(l+1)]/(27-20\delta)$. In the second period a firm serves consumers with $\underline{t}(36-28\delta+9l-8l\delta)/(81-60\delta) \leq t \leq t^{\alpha}(\cdot)$ on segment α and consumers with $\underline{t}(9-8\delta+36l-28l\delta)/(81-60\delta) \leq t \leq 1$ on segment $1-\alpha$, which yields the discounted social welfare over two periods, $SW_4^{1+2}(l,\delta) =$

$$v(1+\delta) - \frac{2}{1-\underline{t}} \int_{\frac{\underline{t}[(9-8\delta)(l+1)]}{27-20\delta}}^{1} \int_0^{1/2} (tx)\,dxdt - \frac{2}{1-\underline{t}} \int_{\underline{t}}^{\frac{\underline{t}[(9-8\delta)(l+1)]}{27-20\delta}} \int_0^{1/2} [t(1-x)]\,dxdt$$

$$-\frac{2\delta}{1-\underline{t}} \int_{\frac{\underline{t}[36-28\delta+9l-8l\delta]}{81-60\delta}}^{\frac{\underline{t}[(9-8\delta)(l+1)]}{27-20\delta}} \int_0^{1/2} (tx)\,dxdt - \frac{2\delta}{1-\underline{t}} \int_{\frac{\underline{t}[9-8\delta+36l-28l\delta]}{81-60\delta}}^{1} \int_0^{1/2} (tx)\,dxdt$$

$$-\frac{2\delta}{1-\underline{t}} \int_{\underline{t}}^{\frac{\underline{t}[36-28\delta+9l-8l\delta]}{81-60\delta}} \int_0^{1/2} [t(1-x)]\,dxdt - \frac{2\delta}{1-\underline{t}} \int_{\frac{\underline{t}[(9-8\delta)(l+1)]}{27-20\delta}}^{\frac{\underline{t}[9-8\delta+36l-28l\delta]}{81-60\delta}} \int_0^{1/2} [t(1-x)]\,dxdt = v(1+\delta)$$

$$-\frac{\underline{t}\left[-l^2\left(-4144\delta^3+6696\delta^2+4455\delta-8019\right)-l\left(512\delta^3-3168\delta^2+5508\delta-2916\right)-10256\delta^3+17\,784\delta^2+8181\delta-18225\right]}{72(27-20\delta)^2(l-1)}.$$

Subtracting $\Pi_4^{1+2}(l,\delta)$ from $SW_4^{1+2}(l,\delta)$ we get the discounted consumer surplus over two periods, $CS_4^{1+2}(l,\delta) = v(1+\delta) +$

$$+\frac{\underline{t}\left[l^2\left(-9584\delta^3+13608\delta^2+157795\delta-22599\right)-l\left(-5632\delta^3+3744\delta^2+20412\delta-20412\right)+4816\delta^3-10872\delta^2+3159\delta+3645\right]}{72(20\delta-27)^2(l-1)}.$$

In the following we analyze how social welfare and consumer surplus change when firms become able to target consumers based on their behavior.

Part 2. i) If $l \leq h_1(\delta)$, then social welfare and consumer surplus do not change with behavior-based targeting.

ii) If $h_1(\delta) < l < 2$, the comparison of social welfare shows that

$$SW_2^{1+2}(l,\delta) - (1+\delta)\,SW_1^{1+2}(l,0) = -\frac{\underline{t}\left[l^2\left(\delta^2+2\delta+1\right)+l\left(2\delta^2+4\delta+2\right)-3\delta^2-10\delta-8\right]}{4(2\delta+3)^2(l-1)}. \qquad (180)$$

The function in the brackets in the nominator of (180) is quadratic in l, opens upwards (for any δ) and has two roots, $h_1(\delta)$ and the negative one (for any δ), such that for any $l > h_1(\delta)$ and δ the right-hand side of (180) is negative. The other comparison shows that

$$CS_2^{1+2}(l,\delta) - (1+\delta)\,CS_1^{1+2}(l,0) = -\frac{\underline{t}\left[l^2\left(2\delta^3+13\delta^2+22\delta+11\right)-l\left(4\delta^3+26\delta^2+52\delta+32\right)+2\delta^3+13\delta^2+28\delta+20\right]}{4(2\delta+3)^2(l-1)}. \qquad (181)$$

The function in the brackets in the nominator of (181) is quadratic in l, opens upwards (for any δ) and has two roots, $h_1(\delta)$ and the other smaller than 1 (for any δ), such that for any $l > h_1(\delta)$ and δ the right-hand side of (181) is negative. We conclude that both social welfare and consumer surplus decrease when firms can recognize consumers.

iii) If $2 \leq l \leq h_2(\delta)$, the comparison of social welfare shows that

$$SW_2^{1+2}(l,\delta) - (1+\delta) SW_3^{1+2}(l,0) = \frac{t\delta\left[l^2\left(4\delta^2+7\delta+3\right)+l\left(8\delta^2+14\delta+6\right)-32\delta^2-101\delta-78\right]}{36(2\delta+3)^2(l-1)}. \quad (182)$$

The function in the brackets in (182) is quadratic in l, opens upwards (for any δ) and has two roots, one of which is negative an the other one is larger than $h_2(\delta)$ for any δ. We conclude that for any $2 \leq l \leq h_2(\delta)$ and any $\delta > 0$ social welfare is smaller with behavioral targeting. The other comparison shows that

$$CS_2^{1+2}(l,\delta) - (1+\delta) CS_3^{1+2}(l,0) = \frac{t\delta\left[l^2\left(26\delta^2+59\delta+33\right)-l\left(92\delta^2+278\delta+204\right)+62\delta^2+203\delta+168\right]}{36(2\delta+3)^2(l-1)}. \quad (183)$$

The function in the brackets of the nominator of (183) is quadratic in l, opens upwards (for any δ) and has two roots, one of which is smaller than 1 and the other one is larger than $h_2(\delta)$ for any δ. It follows that for any $2 \leq l \leq h_2(\delta)$ and any $\delta > 0$ consumers are worse off with behavioral targeting.

iv) If $h_3(\delta) \leq l < 5$, the comparison of social welfare shows that

$$SW_3^{1+2}(l,\delta) - (1+\delta) SW_3^{1+2}(l,0) = \frac{\delta t\left[l^2\left(352\delta^2-1016\delta+918\right)+l\left(-772\delta^2+254\delta+1026\right)-152\delta^2+4987\delta-7182\right]}{36(10\delta-27)^2(l-1)}. \quad (184)$$

The function in the brackets of (184) is quadratic in l, opens upwards (for any δ) and has two roots, one of which is negative (for any δ) and the other one is

$$l_{13}(\delta) := \frac{-\left(-772\delta^2+254\delta+1026\right)+6(27-10\delta)\sqrt{225\delta^2-1016\delta+1045}}{2\left(352\delta^2-1016\delta+918\right)}.$$

Note that $\partial l_{13}(\delta)/\partial\delta < 0$. It also holds that $l_{13}(\delta) = h_3(\delta) \approx 2.28$ if $\delta \approx 0.19$, $l_{13}(\delta) > h_3(\delta)$ if $\delta < 0.19$, with an opposite sign otherwise. Hence, if $\delta > 0.19$, then social welfare increases for any δ and any $h_3(\delta) \leq l < 5$. If $\delta \leq 0.19$, then social welfare increases if $l_{13}(\delta) < l < 5$ and (weakly) decreases otherwise. The other comparison shows that $CS_3^{1+2}(l,\delta) - (1+\delta) CS_3^{1+2}(l,0) =$

$$= \frac{\delta t\left[l^2\left(1472\delta^2-5776\delta+5076\right)-l\left(5732\delta^2-21982\delta+18684\right)+4898\delta^2-17917\delta+14148\right]}{36(10\delta-27)^2(l-1)}. \quad (185)$$

The function in the brackets in (185) is quadratic in l, opens upwards (for any δ) and has two roots one of which is smaller than $h_3(\delta)$ (for any δ) and the other is

$$l_{14}(\delta) := \frac{\left(5732\delta^2-21982\delta+18684\right)+2(27-10\delta)\sqrt{3\left(3347\delta^2-9712\delta+7068\right)}}{2\left(1472\delta^2-5776\delta+5076\right)}.$$

Note that $\partial l_{14}(\delta)/\partial\delta < 0$. It also holds that $l_{14}(\delta) = h_3(\delta) \approx 2.61$ if $\delta \approx 0.48$, $l_{14}(\delta) > h_3(\delta)$ if $\delta < 0.48$, with an opposite sign otherwise. Hence, if $\delta > 0.48$, then consumer surplus increases for any δ and any $h_3(\delta) \leq l < 5$. If $\delta \leq 0.48$, then consumer surplus increases if $l_{14}(\delta) < l < 5$ and (weakly) decreases otherwise.

v) If $5 \leq l \leq \min\{h_4(\delta), h_5(\delta)\}$, the comparison of social welfare shows that

$$SW_3^{1+2}(l,\delta) - (1+\delta)\,SW_4^{1+2}(l,0) = \frac{\delta t\left[l^2\left(352\delta^2-1016\delta+918\right)+l\left(-772\delta^2+254\delta+1026\right)-152\delta^2+4987\delta-7182\right]}{36(10\delta-27)^2(l-1)}. \tag{186}$$

The function in the brackets in the nominator of (186) is quadratic in l, opens upwards (for any δ) and has two roots both of which are smaller than 5 (for any δ). It follows that for any $5 \leq l \leq \min\{h_4(\delta), h_5(\delta)\}$ and any $\delta > 0$ social welfare is larger with behavioral targeting. The other comparison shows that $CS_3^{1+2}(l,\delta) - (1+\delta)\,CS_4^{1+2}(l,0) =$

$$= \frac{t\delta\left[l^2\left(1472\delta^2-5776\delta+5076\right)-l\left(5732\delta^2-21982\delta+18684\right)+4898\delta^2-17917\delta+14148\right]}{36(10\delta-27)^2(l-1)}. \tag{187}$$

The function in the brackets in the nominator of (187) is quadratic in l, opens upwards (for any δ) and has two roots both of which are smaller than 5 (for any δ). It follows that for any $5 \leq l \leq \min\{h_4(\delta), h_5(\delta)\}$ and any $\delta > 0$ consumer surplus is larger with behavioral targeting.

vi) If $l \geq \max\{h_4(\delta), h_5(\delta)\}$, the comparison of social welfare shows that

$$SW_4^{1+2}(l,\delta) - (1+\delta)\,SW_4^{1+2}(l,0) = \frac{t\delta\left[l^2\left(128\delta^2-392\delta+297\right)+l\left(1056\delta^2-2944\delta+2052\right)+128\delta^2-392\delta+297\right]}{36(20\delta-27)^2(l-1)}. \tag{188}$$

The function in the brackets in the nominator of (188) is quadratic in l, opens upwards (for any δ) and has two roots both of which are negative (for any δ), such that for any $l \geq \max\{h_4(\delta), h_5(\delta)\}$ and any $\delta > 0$ social welfare is larger with behavioral targeting. The other comparison shows that

$$CS_4^{1+2}(l,\delta)-(1+\delta)\,CS_4^{1+2}(l,0) = \frac{t\delta\left[l^2\left(1408\delta^2-3736\delta+2457\right)-l\left(2784\delta^2-7648\delta+5292\right)+1408\delta^2-3736\delta+2457\right]}{36(20\delta-27)^2(l-1)}. \tag{189}$$

The function in the brackets in the nominator of (189) is quadratic in l, opens upwards (for any δ) and has two roots both of which are smaller than 5 (for any δ), such that for any $l \geq \max\{h_4(\delta), h_5(\delta)\}$ and any $\delta > 0$ consumer surplus is larger with behavioral targeting.

We now summarize our results for $\delta > 0$. We first combine the results from cases *i)* and *ii)*. Note that $h_1(1) = 1.5$. If $l \leq 1.5$, both social welfare and consumer surplus do not change with targeted pricing. If $1.5 < l < 2$, they decrease for $\delta > h_1^{-1}(\delta)$ and do not change otherwise. We now combine the results from cases *iii)* and *iv)*. Note that $\partial h_2(\delta)/\partial\delta > 0$, $h_2(0) = 2$ and $h_2(1) \approx 2.67$. Hence, if $2 \leq l < 2.28$ ($2 \leq l < 2.61$), then social welfare (consumer surplus) decreases with behavioral targeting for any l and δ. If $2.28 \lessapprox l < 2.29$ ($2.61 \lessapprox l < 2.62$), then social welfare (consumer surplus) (weakly) decrease for $\delta \leq l_{13}^{-1}(\delta)$ ($\delta \leq l_{14}^{-1}(\delta)$), decrease for $\delta \geq h_2^{-1}(\delta)$ and increases otherwise. Finally, if $2.29 \lessapprox l \lessapprox 2.67$ ($2.62 \lessapprox l \lessapprox 2.67$), then social welfare and consumer surplus increase for $\delta \leq h_3^{-1}(\delta)$ and decrease for $\delta \geq h_2^{-1}(\delta)$. From cases *v)* and *vi)* it follows that for any $l > 2.67$ both social welfare and consumer surplus increase for any $\delta > 0$. *Q.E.D.*

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ISSN 2190-9938 (online)
ISBN 978-3-86304-284-4